ELSEVIER

Physica D 97 (1996) 286–290

PHYSICA D

High Rayleigh number convection and passive scalar mixing

Boris I. Shraiman [a], Eric D. Siggia [b,*]

[a] AT & T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974, USA
[b] Cornell University, Laboratory of Atomic and Solid State Physics, Ithaca, NY 14853-2501, USA

Abstract

A brief review is given of recent laboratory investigations of high *Ra* convection and their relation to other turbulent flows. For a passive scalar we summarize an emerging body of theory for the one point distribution function and inertial range correlation functions which display non-Kolmogorov exponents.

Low *Ra* number convection has long been the system of choice for investigating the onset of turbulence in both the small and large aspect ratio limits. For large systems, where there are a multiplicity of possible patterns, experiments might never have gotten started if it were not for the very quantitative road map furnished by Busse and coworkers and its subsequent elaboration and application to large cells via amplitude expansions [1–3]. Theory, as practiced both by physicists and applied mathematicians, truly advanced in step with high quality experiments to sort out what was a very complex problem. Can we look to convection at high *Ra* as a route out of the quagmire of fully developed turbulence?

Someone remarked that turbulence is too important to ignore but too hard to solve; the latter statement being self-fulfilling prophecy in our view. Engineers concern themselves with the first part of the dichotomy and "turbulators" with the second. The sociological aspects of "pure" turbulence research are probably of more interest to a general audience than the scientific issues. The early success of Kolmogorov scaling (K41) seems to have guaranteed the acceptance of its less well-founded successor (K62) [4]. Theory defined what was to be measured; isotropic turbulence was purer than boundary layers, and point measurements plus lots of signal processing were given primacy over visualizations. Large numerical simulations were run whose only output was a spectra. It hardly mattered that fourth-order velocity derivative statistics did not all scale the same way within simulations [5] (and therefore could not be parameterized by fluctuations in the energy transfer rate ϵ) and even isotropic flows were plagued with structures [6]. Engineers, who were not concerned with constructing a general theory of everything, have amassed considerable information about shear flows which brings us back to the subject of convection.

Much effort has gone into the elucidation of the $Nu(Ra)$ relation and potent arguments have been given for scaling according to $Nu \sim Ra^{1/3}$ [7]. Ultimately this relation must fail as Kraichnan long ago recognized when bulk Kolmogorov-like turbulence reached a sufficient Reynolds number so that it generated turbulent boundary layers which invaded the thermal one [8]. No one anticipated however the

* Corresponding author. E-mail: siggia@msc.cornell.edu.

PII S0167-2789(96)00075-9

scaling $Nu \sim Ra^{2/7}$ which Libchaber and coworkers first put on a solid footing [9,10]. In retrospect, exponents indistinguishable from $\frac{2}{7}$ had long been seen in water but it was always assumed that this was a transient on the way to becoming $\frac{1}{3}$ [10]. An essential element for this new scaling relation in our view is the large scale coherent shear flow first seen by Howard and Krishnamurthy [10]. It might seem paradoxical that a wind would lower the $Nu(Ra)$ exponent but in fact the coefficient accompanying the $\frac{2}{7}$ scaling is sufficiently greater than that for marginal stability theory that it overwhelms small differences in exponent.

Our own theory for $\frac{2}{7}$ scaling utilized standard engineering turbulence ideas; energy balance, nesting of the thermal boundary layer within the viscous one, and the kinetic energy dissipation rate for turbulent boundary layers [11]. As a bonus we got a very good fit to the $Re(Ra)$ relation for the mean flow. However $\frac{2}{7}$ scaling extends down to nearly $Ra \sim 10^4$ in aspect ratio ~ 6 cells, and occurs also in 2D convection [10]. In neither circumstance would a conventional turbulent boundary layer occur.

Many visualizations and now more quantitative measurements of the boundary layers in pressured gas cells up to $Ra \sim 10^{11}$ show that plumes are an important component of the near wall turbulence [12,13]. [1] There is at least a qualitative analogy here with wall bounded shear flow where Willmarth and Lu long ago showed that most of the Reynolds' stress is carried by the bursts [14]. Further evidence for the importance of shear in turbulent convection comes from experiments in mercury where the low Pr should enhance the velocities [15]. Here, around $Ra \sim 2 \times 10^7$ a transition to higher Nu was seen which may signal the crossing of the viscous and thermal boundary layers. The surprise is always that scaling arguments that do not admit structures work so well at the two point level. Their success should not blind one to the true nature of the flow.

If turbulent convection reduces to a complicated turbulent boundary layer with local plume forcing, are there any statistical fluid mechanics problems where analytic progress is possible? One possibility is to drop the buoyancy in the Boussinesq equations and look at passive scalar advection.

One small recent success in this area was the prediction and observation of tails in the distribution function (PDF) for the scalar fluctuations in the presence of a mean gradient $\boldsymbol{g}$ (the total temperature field is $\theta - \boldsymbol{g} \cdot \boldsymbol{r}$) [16,17],

$$\partial_t \theta + \boldsymbol{v} \cdot \nabla\theta = \kappa \nabla^2 \theta + \boldsymbol{g} \cdot \boldsymbol{v}. \tag{1}$$

If the velocity field is stationary and homogeneous, $\boldsymbol{g}$ will transmit the same properties to θ. The tails of the θ PDF measure the probability, a parcel of fluid is transported without mixing from a point Δr sufficiently, up/down the gradient such that $g\ \Delta r$ is the desired excursion in θ. If L is the integral scale of v or θ, $\langle\theta^2\rangle \sim L^2 g^2$; so we are looking at transport by the large scales of motion over distances of several or many integral scales. Computation of the tails of PDF is thus equivalent to asking for the fraction of velocity fields in our ensemble which will transport a parcel the desired distance with no mixing. Logically this can occur either for typical mixing rate but an atypical path or for a typical (random walk) path for which the mixing time is anomalously long. The latter effect wins.

Within an integral scale, the probability that the scalar will mix with its environment depends on the shear of the large scales either through its direct action or in its role in maintaining the turbulent energy cascade and through it the eddy diffusivity. The strain is $\lesssim 1/T$ in magnitude ($T \gg 1$ in units where $\langle v^2\rangle/L^2 \sim 1$) with probability $\sim (1/T)^a$. Impose this condition T time in succession with a probability $\mathrm{e}^{-a\ T\ \ln\ T}$. If there were no diffusion, the PDF from fluid parcels random walking up and down the gradient for a time T would be non-stationary and of the form $\exp(-\theta^2/2T)$ in suitable units. Therefore the tails are given by

$$P(\theta) \sim \int \mathrm{d}T \mathrm{e}^{-\theta^2/2T} \mathrm{e}^{-aT\ \ln\ T} \sim \mathrm{e}^{-|\theta|\ \ln\ |\theta|}. \tag{2}$$

[1] We do not agree that the maximum cutoff frequency in the temporal spectrum of the scalar is a surrogate for the large scale velocity, its dependence on Ra may merely track the turbulent homogenization of the scalar.

A proper calculation is very naturally formulated as a path integral over the prior history of the parcel and eliminates the $\ln|\theta|$ in (2) [17].

A related question with relevance to the treatment of small scale θ statistics is to examine the PDF of $\nabla\theta$. Within simulations [18], the distribution is cusped around its maximum and has stretched exponential tails. However the cusp is centered around $-\boldsymbol{g}$. This is manifest in snapshots of the total temperature field, as plateaus where the temperature is uniform, separated by "cliffs". The mean gradient is thus expelled into the cliffs. Interestingly enough, the path integral calculation of Pumir et al. [17], which had only a single scale velocity, did reproduce this feature of the simulations.

A more ambitious problem is to look within an integral scale and study the scaling of θ fluctuations advected by a velocity which itself scales. This is an appealing problem since all the Kolmogorov phenomenology carries over to the scalar. The spectrum of θ often follows a $\frac{5}{3}$ law though with less precision than for the velocity [19].

The surprising fact is that Kolmogorov theory fails badly already for the third-order moment which is frequently packaged as $S_d = \langle(\boldsymbol{g}\cdot\partial\theta)^3\rangle/\langle(\partial\theta)^2\rangle^{3/2}$. The small scales should be isotropic irrespective of large scale gradients so we expect $S_d \sim g/\langle(\partial\theta)^2\rangle^{1/2} \sim Re^{-1/2}$. Instead the experiment very unambiguously says S_d is Re independent and ~ 1 [19]. Additional encouragement to tackle this problem analytically came from simulations and wind tunnel experiments which showed that the anomalous skewness was not unique to shear flows but true even for synthetic Gaussian velocity fields [19]! The skewness is symptomatic of coherent structures in the θ field consisting of gentle ramps and abrupt cliffs (noted already above) across which θ may fall by a good fraction of its variance over a microscale in distance [18,19]. This is qualitatively the same property that was noticed for the coherent vortex tubes seen in isotropic simulations [5]. A velocity difference of order the RMS occurs across the vortex core which itself scales with the Kolmogorov length. Kraichnan has shown by a plausible closure and simulations that even powers of $\Delta\theta_r = (\theta(r) - \theta(0))$ exhibit non-Kolmogorov scaling when v is Gaussian white noise with a 5/3 spectrum [20].

The method of choice for calculating higher-order θ correlation functions is the Hopf equation which expresses their stationary. This equation is simple only for δ-correlated velocity fields where it reduces to the sum of the Richardson operator acting on all pairs of points (a, b are spacial labels).

$$L_{\mathrm{R}}^{(\zeta)} = r_{ij}^{2-\zeta}\left(\delta^{ab}(d+1-\zeta) - (2-\zeta)\hat{r}_{ij}^{a}\hat{r}_{ij}^{b}\right)\partial_i^a\partial_j^b. \tag{3}$$

Lower-order correlators act as source terms for higher-order ones, and the inhomogeneous solutions for this hierarchy at least formally follow Kolmogorov scaling [17]. Several groups realized that new non-trivial exponents can arise via homogeneous solutions [21,22].

Our own approach has been to eschew a white noise model in favor of a more phenomenological treatment of a velocity field with proper temporal correlations [22]. The Hopf equation we ultimately solve is only approximate but in the end more realistic. We summarize the salient features of our solution.

The Navier–Stokes equations have no intrinsic scales except for viscosity so that the action of δv_r for one Lagrangian correlation time amounts to a change in δr by O(1). Therefore given a multipoint correlation function whose points have a radius of gyration R_{g} decompose the velocity into scales $\gg R_{\mathrm{g}}$ which contribute only through their action on the mean temperature gradient, $\boldsymbol{g}\cdot\boldsymbol{r}$; scales $\sim R_{\mathrm{g}}$ whose affect we approximate by a volume preserving linear transform acting coherently on all points, and scales $\ll R_{\mathrm{g}}$ (but scaling with R_{g}) which act as an incoherent nearly white noise random advection.

The evolution of the correlation function under the coherent part of the velocity is expressed by $L_0 = L_{\mathrm{R}}^{(0)}$ and the incoherent part by some L_{D} (D – for dissipation) to be defined below. The stationarity dictates the Hopf equation:

$$(L_0 + L_D)\langle\theta(1)\cdots\theta(n+1)\rangle = I, \tag{4}$$

where the RHS denotes the inhomogeneous forcing term. It can be argued that the inhomogeneous solution of (4) has the regular (K41) scaling e.g. $R_{\mathrm{g}}^{(n+1)/3}$ for even moments and $R_{\mathrm{g}}^{(n+3)/3}$ for odd. The scaling of the homogenous solutions, which of course are needed to satisfy the matching or boundary condition for large and small R_g, cannot be found by dimensional analysis. Hence we need to determine the scaling index, λ, of the zero modes of the $L_0 + L_D$ operator. The anomalous exponents correspond to those λ less than the corresponding K41 exponent.

Let us start with the zero modes of L_0. For the $(n+1)$ point correlation function of r_i eliminate the center of mass by defining n difference vectors, $\rho_1 = (r_1 - r_2)/\sqrt{2}$, $\rho_2 = (r_1 + r_2 - 2r_3)/\sqrt{6}$, $\rho_3 = (r_1 + r_2 + r_3 - 3r_4)/\sqrt{12}$, etc. Since the coherent part of the velocity field is modeled as a single matrix acting on all ρ_i, one observes that this dynamics and hence L_0 are invariant under all volume preserving linear transformations $\boldsymbol{\rho}_i \to g_{ij} \cdot \boldsymbol{\rho}_j$ acting on the "isospin" label of ρ's. These form the $SL(n, R)$ groups. In addition L_0, is invariant under spatial rotations, $SO(d)$, and a dilation $\wedge = \rho_i^a \partial_i^a$. Since L_0 is second order in derivatives, the above-mentioned invariances imply that it can be expressed entirely in terms of their Casimir operations (i.e. angular momentum squared for $SO(d)$ and an appropriate generalization thereof for $SL(n)$ [22].

These group theortic considerations allow one to immediately determine all the eigenmodes of L_0. In the simplest case of the skewness in two dimensions, they can be labeled by an angular momentum quantum number l, ν related to the $SL(n=2)$ Casimir, q an $SL(2)$ "angular momentum", and finally the dilation exponent λ. Demanding zero eigenvalue and imposing boundary conditions leads to $\lambda = 1$, $\nu = \frac{1}{2}$, $l = 1$, and arbitrary q (plus other less relevant λ's). Note $\lambda = 1$ is precisely what is needed to explain the anomalous skewness and gives one hope that L_D will indeed be a small perturbation on L_0.

Before considering L_D we note another important physical property of L_0 namely its reducibility; acting on any function behaving as $|r_{ij}|^x$, $x > 0$; the limit $r_{ij} \to 0$ reduces $L_0(n)$ to $L_0(n-1)$ acting on the remaining variables. This is what one would expect for the advection.

This reducibility must not hold for the dissipative operator L_D which is most plausibly taken as $\alpha R_g^{2/3} L_R^{2/3}$, where α will be used as a formal expansion parameter, $R_g^{2/3}$ adjusts the time scale to that assumed for L_0 (alternatively L_D must have the same scaling dimensions as L_0), and $L_R^{2/3}$ is just the operator appropriate to a white noise velocity with Kolmogorov spectrum. For technical reasons we use a simpler variant with the same scaling properties $L_D = \alpha R_g^{2/3} \beta(\rho) \nabla_\rho^2$, where β has a scaling dimension $\frac{4}{3}$, vanishes as $r_{ij}^{4/3}$ when any $r_{ij} \to 0$ (a "Kolmogorov point"). Near such a point the correlation function is isotropic in $\boldsymbol{r}_{ij}$, and behaves like $|r_{ij}|^{2/3}$.

The $L_0 + L_D$ Hopf operator, which is ultimately phenomenological, has many intuitively reasonable properties in addition to those already noted for L_0 and L_D separately. Exponents are non-universal in that they depend on details of the velocity field in addition to its scaling dimension. If $\rho_1 = r_{12} \to 0$ then

$$L_0^{(n-1)} \langle \theta^2(1)\theta(2)\ldots\rangle + \alpha R_g^{2/3} \langle \epsilon(1)\theta(2)\ldots\rangle = 0. \tag{5}$$

The new operator ϵ is the local dissipation rate, expressible as $L_R^{(2/3)}(\rho)\theta(\rho+1)\theta(1)$ in the inertial range and matching to $\kappa(\nabla\theta)^2$ in the dissipation range. Of course (5) is not a closed equation, but $\langle \epsilon\theta\ldots\rangle$ must have a positive scaling dimension, ((1) implies $\sup_r |\theta(r)|$ and $\langle\epsilon\rangle$ are set by the large scales). Thus the exponent for $\langle\theta\theta\ldots\rangle$ must exceed $\frac{2}{3}$.

The dissipative term is a singular perturbation in (4) just as it is in (1). It dominates near the Kolmogorov points and also when several points in the correlations are parallel. Collinearity is of course an invariant condition under matrix multiplication, functions invariant under L_0 alone are non smooth there and inclusion of the incoherent eddy damping renders them analytic.

The perturbative calculation in L_D thus requires dealing with the non-trival crossover from the L_0 dominated region to the L_D dominated one, which for the skewness in two or three dimensions occurs for $\zeta/R_g^2 \sim \alpha^{1/2}$ with the "volume" $\zeta = |\rho_1 \wedge \rho_2|$ (note that collinearity corresponds to $\zeta = 0$). This calculation has been described in [22]. Its result, aside from the scaling exponent $\lambda(\alpha) = 1 + 0(\alpha)$, is the explicit form of the correlator for a general configuration of points. Away from collinearity this can be represented as a linear superposition of the L_0 zero modes, ψ, forming a representation of $SL(n=2)$:

$$\langle\theta\theta\theta\rangle = \sum_q a_q \psi_{\nu=\lambda/2,l=1,q}(\rho). \qquad (6)$$

The coefficient a_q is related to this correlator with points collinear. Thus a_q can be input from an experimental measurement and the correlation function determined for non-collinear points. A more compact integral form for (6) is given in [22]. How accurate this procedure is, must be decided by experiment [22].

We close with a few remarks about the velocity. Recently very convincing evidence has been given for the small scale isotropy of the two point velocity correlations [23]. One should not conclude that higher-order correlations are isotropic. Motivated by analogy with the scalar skewness Pumir and Shraiman [24,25] studied homogeneous shear ($\langle \boldsymbol{v}\rangle \alpha \; y\hat{x}$) numerically and found that both ω_z and $\partial_y v_x$ had a normalized third moment $\sim$ O(1) and Re independent. Kolmogorov theory predicts again $Re^{-1/2}$. Should the trend seen only for $R_\lambda \lesssim 100$ numerically persist to much higher Re, we will again have a clear violation of K41 at the level of a 3-point function. Shear flows may thus prove to be a better idealization of turbulence than isotropic ones because they explicitly include a steady large scale shear which biases the inertial range statistics all the way down to dissipative scales.

It is only in the context of a Festschrift that the authors could pack so many speculations and suppositions into a volume that carries Fritz Busse's name. May his comments and writings keep us on our mettle for many years to come.

References

[1] F.H. Busse, in: Topics in Applied Physics, eds. H.L. Swinney and J. Gollub, Vol. 45 (Springer, Berlin, 1981) p. 97.
[2] A. Newell, T. Passot and J. Lega, Ann. Rev. Fluid Mech. 25 (1993) 399.
[3] M.C. Cross and P.C. Hohenberg, Rev. Modern Phys. 65 (1993) 851.
[4] S. Chen et al. Phys. Rev. Lett. 74 (1995) 1755.
[5] E.D. Siggia and J. Fluid Mech. 107 (1981) 375; W.T. Ashurst, A.R. Kerstein, R.M. Kerr and C.H. Gibson, Phys. Fluids 30 (1987) 3243; J. Jimenez, A.A. Wray, P.G. Saffman and R.S. Rogallo, J. Fluid Mech. 255 (1993) 65.
[6] S. Douady, Y. Couder and M.F. Brachet, Phys. Rev. Lett. 67 (1991) 983.
[7] E.A. Spiegel, Ann. Rev. Astron. Astrophys. 9 (1971) 323.
[8] R.H. Kraichnan, Phys. Fluids 5 (1962) 1374.
[9] B. Castaing et al. J. Fluid Mech. 204 (1989) 1.
[10] E.D. Siggia, Ann. Rev. Fluid Mech. 26 (1994) 137.
[11] B.I. Shraiman and E.D. Siggia, Phys. Rev. A 42 (1990) 3650.
[12] A. Belmonte, A. Tilgner and A. Libchaber, Phys. Rev. Lett. 28 (1993) 4067.
[13] A. Belmonte and A. Libchaber, Phys. Rev. E, to appear.
[14] S.S. Lu and W.W. Wilmarth, J. Fluid Mech. 60 (1973) 481.
[15] S. Cione, S. Ciliberto and J. Sommeria, Dynamic Atmos. and Oceans 24 (1996) 117.
[16] J. P. Gollub et al. Phys. Rev. Lett. 67 (1991) 3507; Jayesh and Z. Warhaft, Phys. Rev. Lett. 67 (1991) 3503.
[17] A. Pumir, B.I. Shraiman and E.D. Siggia, Phys. Rev. Lett. 66 (1991) 2984; B.I. Shraiman and E.D. Siggia, Phys. Rev. E 49 (1994) 2912.
[18] M. Holzer and E.D. Siggia, Phys. Fluids 6 (1994) 1820; A. Pumir, Phys. Fluids 6 (1994) 2118; C. Tong and Z. Warhaft, Phys. Fluids 6 (1994) 2165.
[19] P.G. Mestayer and J. Fluid Mech. 125 (1982) 475; K.R. Sreenivasan, Proc. Roy. Soc. London 434 (1991) 165.
[20] R.H. Kraichnan, Phys. Rev. Lett. 72 (1994) 1016.
[21] M. Chertkov et al. Phys. Rev. E 51 (1995) 5609; K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. 75 (1995) 3834.
[22] B.I. Shraiman and E.D. Siggia, C.R. Acad. Sci. Paris 321 (1995) 279; Phys. Rev. Lett., submitted.
[23] S.S. Saddoughi and S.V. Veeravalli, J. Fluid Mech. 268 (1994) 333.
[24] A. Pumir and B.I. Shraiman, Phys. Rev. Lett. 75 (1995) 3114.
[25] M.J. Lee, J. Kim and P. Moin, J. Fluid Mech. 216 (1990) 561; J.L. Balint, J.M. Wallace and P. Vakoslavcevic, J. Fluid Mech. 228 (1991) 53.