Symmetry and Scaling of Turbulent Mixing

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The stationary condition (Hopf equation) for the $(n+1)$-point correlation function of a passive scalar advected by turbulent flow is argued to have an approximate $\mathrm{SL}(n,R)$ symmetry which provides a starting point for our phenomenological theory in which less symmetric terms are treated perturbatively. The large scale anisotropy is found to be a relevant field, in contradiction with Kolmogorov phenomenology, but in agreement with the large scalar skewness observed in shear flows. Exponents are not universal, yet quantitative predictions for experiments to test the $\mathrm{SL}(n,R)$ symmetry can be formulated in terms of the correlation functions. [S0031-9007(96)01137-4]

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Kolmogorov succeeded, by very simple arguments (K41), in predicting the scaling for the velocity correlations $\langle v_k v_{-k}\rangle$ at high Reynolds number R, for wave numbers intermediate between those defined by the geometry of the flow and dissipation; the so called inertial range [1]. Obukhov and Corrsin soon applied the K41 reasoning to a passive scalar, i.e., a field Θ obeying

$$\partial_t\Theta + \vec{v}\cdot\vec{\nabla}\Theta = \kappa\nabla^2\Theta \qquad (1)$$

(κ is diffusivity), and experiments subsequently found scaling behavior for Θ, although the K41 exponent is approached only at very high R if at all [2]. However, an even more glaring inconsistency with K41 appeared with the observation that the derivative skewness $s_d = \langle(\partial_x\theta)^3\rangle/\langle(\partial_x\theta)^2\rangle^{3/2}$ is of order 1 and R independent out to the highest R available [3]. This fact motivated our work.

Since s_d breaks parity, we follow the experiments and impose a large scale gradient $\vec{g}$ so that $\Theta = \theta + gr$ where $\langle\theta\rangle = 0$ and $\langle\theta(r)\theta(0)\rangle$ inherits a correlation length or integral scale from the velocity field. The inhomogeneous term $\vec{g}\cdot\vec{v}$ which appears in the θ equation then acts as a "force" which maintains the θ fluctuations stationary. The conundrum with s_d is that K41 predicts small scale isotropy for large R specifically, $s_d \sim g/\langle(\partial_x\theta)^2\rangle^{1/2} \sim R^{-1/2}$ or for $\delta\theta_r = \theta(r) - \theta(0)$, $S = \langle\delta\theta_r^3\rangle \sim r^{5/3}$ (vs r^1 in experiments). The K41 arguments have proved so seductive and work so well for the two-point velocity correlations that problems with the scalar have been all but forgotten.

The observation of $s_d \sim O(1)$ is particularly intriguing since it suggests that $\vec{g}$ is a *relevant* variable along with the "energy" (scalar variance really) dissipation rate ϵ_θ, in the sense of parametrizing the effect of the large, geometry specific scales on the inertial range. Averaging $\vec{g}$ away by insisting on isotropic large scales will eliminate s_d but in no way lessen the instantaneous nonlocal interactions between large and inertial range scales. This merely obscures the relevant physics.

The prospects for an analytic theory of scalar intermittency are much enhanced by the observation that s_d remains ~ 1 and R independent even when the complex turbulent shear flow is replaced by a Gaussian v field [4]. "Boring" but scale invariant velocity fields generate interesting scalar statistics; thus one expects that much can be learned from studying a passive scalar advected by Gaussian random velocity—the so called Kraichnan model [5] and its extension to more physical nonwhite temporal correlations.

In this Letter we argue that the dominant term in the evolution equation (the "Hopf equation") of the scalar multipoint correlator ψ generated by a velocity field with a physical correlation time is highly symmetric and is integrable by Lie algebraic methods. Recently it has been realized [6–8] that the leading (anomalous scaling) terms in ψ arise as the zero modes of the Hopf operator L_H with the lowest scaling index λ. For the dominant, symmetric part of L_H, we find that the complete set of zero modes has an infinite degeneracy (because of the high symmetry), i.e., λ is independent of a subset $\{q\}$ of the quantum numbers. The degeneracy is lifted by the lower symmetry part of L_H which dominates when points are collinear and is treated by singular perturbation theory, or equivalently by diagonalizing within the degenerate subspace. The correlator is no more universal than the velocity field. However, since ψ with a general configuration of points is governed approximately by the symmetric part of L_H, it can be represented as a linear superposition of the degenerate $\{q\}$ modes with the weight defined by ψ with points collinear. The latter can be measured directly in experiment or numerical simulations and the above relation used as a quantitative check for the existence of the $\mathrm{SL}(n,R)$ symmetry.

The homogeneous part of the inertial range Hopf equation for a white noise velocity field may be written in terms of the Richardson operator

$$L_R^{(\zeta)} = r_{ij}^{2-\zeta}(\delta^{ab}(d + 1 - \zeta) - (2 - \zeta)\hat{r}_{ij}^a\hat{r}_{ij}^b)\partial_i^a\partial_j^b \tag{2}$$

acting on all pairs of points $(i, j, \vec{r}_{ij} = \vec{r}_i - \vec{r}_j)$ and where a, b denote spacial indices [9–11]. In this paper we explore the consequences of a complementary class of models in which the proper scaling of the Lagrangian time scale τ with r, $\sim r^{2/3}\epsilon^{-1/3}$, is made paramount. The price paid is that an exact, temporally local Hopf equation does not exist and phenomenological arguments become necessary.

Consider a multipoint correlation function whose arguments are roughly equally spaced, with radius of gyration R_g, which partitions the modes of the velocity field into three bands by scale size. Relative motion of the $\vec{r}_i$ is predominantly caused by modes with scale $\sim R_g$ which act like a single coherent strain-vorticity matrix whose effect over one correlation time is an $O(1)$ volume preserving linear transformation. For Gaussian $\vec{v}$, the averaged operator implementing this finite coordinate change is just the exponential of $L_0 = L_R^{(0)}$, so the condition of stationarity can be expressed as $L_0\psi = 0$. (Note L_0 is identical with the Batchelor-Kraichnan operator [10] for white random gradient advection but its meaning here is quite different.) This approximation clearly fails when two points, e.g., (1) and (2), approach $r_{12} \ll R_g$. Then $\vec{r}_{12}$ is acted on by the relative velocity from its own scale whose effects exceed those of the coherent strain vorticity and which is white in comparison. This "eddy damping" we will take account of below by adding a dissipative term L_D to L_0. Finally, the velocity on scales $\gg R_g$ is an overall translation, which generates the inhomogeneous term in the Hopf equation in the presence of a mean gradient, but does not affect the zero modes.

Our phenomenological description or indeed any Hopf equation more elaborate than white noise will contain parameters beyond ζ and d in (2) which originate from the velocity field. In general, one should expect the anomalous exponents to depend on all parameters of the Hopf equation and thus be nonuniversal. For this reason we concentrate in the remainder of the paper on analyzing the consequences for ψ of a symmetry, exact for L_0, and approximate for the full Hopf operator.

The operator L_0 is a very attractive starting point for perturbation theory since it is integrable [6]. For the $(n + 1)$ point correlation function of r_i eliminate the center of mass by defining n difference vectors $\rho_1 = (r_1 - r_2)/\sqrt{2}$, $\rho_2 = (r_1 + r_2 - 2r_3)/\sqrt{6}$, $\rho_3 = (r_1 + r_2 + r_3 - 3r_4)/\sqrt{12}$, etc.

$$L_0(n) = -(d + 1)L^2 + 2dG^2 + d(d - n)(\Lambda^2/nd + \Lambda), \tag{3}$$

where $L^2 = \frac{-1}{2}\sum_{a,b}(\rho_i^a\partial_i^b - \rho_i^b\partial_i^a)^2$ is the square of the total angular momentum; $\Lambda = \rho_i^a\partial_i^a$ is the dilatation operator; and $G^2 = \frac{1}{2}\sum G_{ij}G_{ji}$, $G_{ij} = \rho_i^a\partial_j^a - \frac{1}{n}\delta_{ij}\Lambda$ is the Casimir of the group $SL(n, R)$ diagonal in space, acting on the i label (henceforth called pseudospace) via $\rho_i^a \to g_{i,j}\rho_j^a$, where g is a unit-determinant matrix. The appearance of the $SL(n, R)$ in (3) is the consequence of the underlying evolution step being the multiplication of all $\vec{\rho}_i$ by a common spatial strain-vorticity matrix—the linear mapping described earlier—which is invariant with respect to a general linear transformation acting on ρ's. The last three terms in (3) can be rewritten (by rearranging the summations in G^2 to contract ρ and ∂ first on pseudospace and then on space) as $2dJ^2$ where J^2 is the Casimir of $SL(d, R)$ group acting on real (rather than pseudo) space. However, the generators of $SL(d, R)$ do not commute with L^2 so except in special cases it is more convenient to organize the zero modes of L_0 according to the representations of $SO(d) \times SL(n) \times \Lambda$ which will simultaneously diagonalize the three operators in (3). Before constructing these representations for $n = 2, d = 2, 3$ ($n = 3, d = 2$ is similar) we note several general properties of the eigenvalues.

We will work within the space of homogeneous functions whose scaling dimension λ diagonalizes Λ. The other quantum numbers which label the correlators ψ are discrete since the remaining variables are compact. To solve for λ note that $L^2 \to l(d - 2 + l)$ and $G^2 \to (n - 1)(1 + \lambda/n)\lambda/2$; the later expression comes from the fact that the spectrum of G^2 does not depend on d and hence can be evaluated by relating G^2 to J^2 and setting $d = 1$ or $J^2 = 0$. (An additional discrete quantum number k can enter G^2 but is not relevant at this stage [6].) Thus for the zero modes, (3) reduces to a quadratic equation in λ, and the most relevant (i.e., smallest non-negative) [12] eigenvalues for any (n, d) are $\lambda = (0, 1)$ for $\ell = (0, 1)$, respectively. The former pertains to flatness ($n = 3$) and the latter to skewness since it must couple to the external gradient. The $\ell = 0$ eigenvalue could have been anticipated from the Batchelor-Kraichnan interpretation of L_0.

It is both convenient computationally and illuminating of the Lie algebraic structure to construct the zero modes of (3) in integral form, which we will do explicitly for $n = 2$, $d = 2, 3$. First, for $n = d = 2$ consider a homogeneous function of degree 2ν:

$$h_{q,m}^{\nu}(\rho) = \int d\phi \int d\psi e^{iq\phi + im\psi}[n_i(\phi)\rho_i^a e_a(\psi)]^{2\nu}, \tag{4}$$

where integration over the r-space and p-space unit vectors $\hat{e}(\psi)$ and $\hat{n}(\phi)$, respectively, ensures that h^ν has r-space angular momentum ℓ and p-space angular momentum [eigenvalue of $\frac{i}{2}(G_{12} - G_{21})$] is q. It is easy to verify: $G^2 h_{q,m}^{\nu} = \nu(\nu + 1)h_{q,m}^{\nu}$. It can be shown that $h_{q,m}^{\nu}$ transforms linearly under the action of the $SL(2, R)$ transformations forming an infinite dimensional representation [13]. Instead we will use similar algebraic manipulations to evaluate (4) in terms of the Euler parametrization of ρ: $\rho_i^a = \sum_{j=1}^2 R_{ij}(\chi)\xi_j R_{ja}(\theta)$ where $R(\chi), R(\theta)$ are rotation matrices (which can be obtained by diagonalization of $\rho\rho^T$). Substituting into (4), we observe that the R matrix factors can be absorbed into a redefinition of the integration variable resulting in a $\exp(iq\chi + im\theta)$ phase factor. Next rescale $n_i\xi_i$ to define a new unit vector $\hat{n}_i = n_i\xi_i/\sqrt{n_1^2\xi_1^2 + n_2^2\xi_2^2}$ and its phase $\exp[i\delta(\phi,\xi)] = \hat{n}_1 + i\hat{n}_2$. All the ψ dependence of (4) is in the form $\int e^{im\psi}\cos^{2\nu}(\delta + \psi)$, so to within a constant

$$h_{q,m}^{\nu} = e^{iq\chi + im\theta}\int_0^{2\pi}\frac{d\theta}{2\pi}\, e^{iq\phi}(n_1^2\xi_1^2 + n_2^2\xi_2^2)^{\nu} \times \left(\frac{n_1\xi_1 - in_2\xi_2}{n_1\xi_1 + in_2\xi_2}\right)^{m/2} \equiv e^{iq\chi + im\theta}(\xi_1\xi_2)^{\nu}P_{qm}^{\nu}(w^{-1}), \quad (5)$$

where $w = 2\xi_1\xi_2/(\xi_1^2 + \xi_2^2)$ and the integral defines the Jacobi function P^{ν}.

To complete the construction of the eigenfunctions, note that the area of the triangle $\det\rho = \vec{\rho}_1 \wedge \vec{\rho}_2 = \xi_1\xi_2$ is invariant under both SO(2) and SL(2) transformations. The complete set of eigenfunctions with four quantum numbers is $\psi_{kqm}^{\lambda} = (\det\rho)^{\lambda/2}h_{qm}^{(\lambda/2+k)}$ where $k = \nu - \frac{\lambda}{2}$ is a positive integer to ensure analyticity as $\det\rho \to 0$ and points in the correlator become collinear.

To generalize to $d = 3$, replace $R_{ij}(\theta)$ in the expression for ρ_i^{ν} by a pair of orthogonal unit vectors $\hat{\eta}_{1,2}$ defining a representation of SO(3) (with $\hat{\eta}_3 = \hat{\eta}_1 \wedge \hat{\eta}_2$ completing the triad). Let the 3D unit vector $\hat{e}(\psi)$ in (4) be perpendicular to and rotate around $\hat{\eta}_3 \parallel \vec{\rho}_1 \wedge \vec{\rho}_2$ which one observes is invariant under SL(2) transformations. Repeating the earlier derivation and observing h^{ν} can be multiplied by an arbitrary function of $\vec{\rho}_1 \wedge \vec{\rho}_2$, e.g., $Y_m^{\ell}(\hat{\eta}_3)$ one arrives at the eigenvectors

$$\psi_{\nu,q,l,m,m'}^{\lambda} = (\xi_1\xi_2)^{\lambda/2}P_{q,m'}^{\nu}(w^{-1})D_{m,m'}^{l}(\eta)e^{iq\chi}, \quad (6)$$

which have a "complete" set of $nd = 6$ quantum numbers (m' is not summed).

For $\ell = 1$, D^{ℓ} in (6) reduces to either $\hat{\eta}_1$ or $\hat{\eta}_2$ (η_3 is excluded for the skewness by inversion symmetry), and $k = 0$ for the relevant λ; thus the manifold of degenerate states with $\lambda = 1$ is labeled by the quantum number q or the angle χ. To proceed further we must include the effects of the small scale, incoherent velocity, via the operator L_D and write the Hopf equation as $L_H\psi = (L_0 + L_D)\psi = 0$. We first discuss properties of L_D that pertain to any (n,d) and then specialize to the skewness [14].

The operator L_0 is reducible, that is, acting on any function behaving as $|r_{ij}|^x$, $x > 0$ the operator $L_0(n)$ reduces in the limit $r_{ij} \to 0$ smoothly to $L_0(n-1)$ acting on the remaining variables. This is because L_0 originates from the advective part of (1) which evolves $\theta^2(r)$ the same way as $\theta(r)$. The operator L_D incorporating the dissipation cannot reduce so simply and further must lower the continuous pseudospace symmetry of L_0 to the discrete permutations of the correlator. The most plausible form for L_D is $\alpha R_g^{2/3}L_R^{(2/3)}$, i.e., just the Hopf operator for the white velocity model with K41 exponents, multiplied by the Lagrangian time scale $R_g^{2/3}$ implicit in L_0, with a formally small expansion parameter α. Alternatively L_D should have the same scaling dimension as L_0; they both represent inertial range dynamics.

When two points coalesce, $\rho = r_{ij} \to 0$, zero modes of L_H behave as $f_{\lambda}^{(1)} + f_{\lambda-2/3}^{(2)}\rho^{2/3}$ where $f^{1,2}$ are functions of the other coordinate differences with the indicated degree. The leading balance in the equations is $L_0 f^1 + f^2 L_D\rho^{2/3} = 0$, and L_D is a singular perturbation to L_0. If $L_R^{2/3}(\rho)\theta(\rho + 1)\theta(1)$ is identified with the local dissipation rate ε, then the same balance reads, as $\rho \to 0$,

$$L_0^{(n-1)}\langle\theta^2(1)\theta(2)\ldots\rangle + \alpha R_g^{2/3}\langle\epsilon(1)\theta(2)\ldots\rangle = 0 \quad (7)$$

Of course (7) is not a closed equation, but it does correctly reexpress a consequence of scaling and the behavior of correlators with nearly coincident points, namely that ε carries a scaling dimension of $2/3$ relative to θ^2. Phenomenology suggests any L_D which forces the correct local singularity in ψ for $\rho \ll R_g$ should suffice. The Laplacian damping model of Ref. [6], $L_D = \frac{\alpha}{2n}R_g^2\nabla^2$, involved derivatives in all ρ_i^a symmetrically and therefore left correlators analytic when points tended to coincidence.

The eigenvalue problem for the skewness zero modes, ψ_3, can be reduced to the χ, w plane with the symmetries of antiperiodicity under $\chi \to \chi + \pi/3$ (permutation and reflection of r_i) and even under $(\chi,\xi_1) \to -(\chi,\xi_1)$, $\xi_2, \vec{\eta}_i$ unchanged (interchange of $r_{1,2}$. Away from $w = 0$ where L_D dominates, ψ_3 is a combination of (6) with $q = \pm 3, \pm 9$, etc. and $l = 1$.

To understand which features of the skewness zero modes are independent of the details of L_D and thus of experimental interest, several salient aspects of the singular perturbation theory for L_H must be recounted. A crossover equation has been solved which takes the "outer" solution (L_0 dominant) for $w \lesssim 1$, and χ arbitrary, and propagates it inward to the $w = 0$ line where L_D dominates. Then a closed eigenvalue equation in χ alone is derived, which is solved by shooting from $w = 0, \chi = 0$ to $\chi = 2\pi/3$. The L_D dependent eigenfunction $a(\chi)$ along $w = 0$ determines the combination of q modes that are superimposed for $w > 0$ in the solution. However, a feature of the crossover equation for our class of models is that to first nontrivial order in perturbation theory, the relation of $a(\chi)$ to the amplitudes of the SL(n) modes (6) in the outer solution is independent of L_D. Hence if one were to measure the skewness with all points parallel (i.e., $w = 0$), $a(\chi)$ would be known and from it the skewness for $w \lesssim \mathcal{O}(1)$ determined using only the SL(n) symmetry of L_0.

This insight can be most compactly expressed by using the integral form (4) for the SL(n) modes (5) and replacing the sum on q by a convolution integral on the pseudospace angle; $\psi_3 \equiv \langle \theta_1 \theta_2 \theta_3 \rangle$ becomes with $\nu = \lambda/2$,

$$\psi_3(\rho) = \vec{g} \cdot \int_0^{\pi} \frac{\cos(\phi)\xi_1 \vec{\eta}_1 + \sin(\phi)\xi_2 \vec{\eta}_2}{[\cos^2(\phi)\xi_1^2 + \sin^2(\phi)\xi_2^2]^{1/2-\lambda/2}} \times f(\phi - \chi)\, d\phi\,, \tag{8}$$

The modes (5) were summed on m' to impose the symmetry under $r_{1,2}$ interchange. The crossover equation relates the L_D dependent weight $f_q = \mathrm{sgn}(q)\,(q^2 - 1)a_q$ to the eigenfunction $a(\chi)$. Note ψ_3 for $\vec{\rho}_i$ parallel [$\xi_1 = 0$ and $\tan(\chi) = \rho_1/\rho_2$] is just $\xi_2^{\lambda} a(\chi)\vec{\eta}_2 \cdot \vec{g}$ to within an overall constant. The symmetries of ψ_3 and K41 imply $f(\varphi)$ should have the same symmetry, singularities as $\sin(3\varphi)/|\sin 3\varphi|^{4/3+2\nu}$(N.B. $2\nu \sim 1$, the integral exists).

Equation (8) can readily be tested in a wind tunnel with several probes along the mean gradient in y and time lags to sample x.

The four-point function is more influenced by L_D than is ψ_3 judging both by inequalities on its exponent derived below, and explicit calculations [e.g., corrections to the $L_D = 0$ exponents begin as $\mathcal{O}(\alpha)$ for the skewness but $\mathcal{O}(\alpha^{1/2})$ for the flatness]. For decaying turbulence the structure function $\langle [\theta(1) - \theta(0)]^n \rangle \leq (2\sup_x |\theta|)^{n-2} \langle [\theta(1) - \theta(0)]^2 \rangle$ since advection diffusion does not increase the maximum of θ. It is physically plausible that the small scales would be unaffected if we forced and assumed $\sup_x |\theta|^2 \sim \langle \theta^2 \rangle$. Thus all positive anomalous exponents for ψ_4 must *exceed* $\frac{2}{3}$ assuming K41. By starting from the strong coupling limit $L_0 \psi_4 = 0$ we found $\lambda = 0$ which has to be renormalized up to $\frac{2}{3}$ by L_D. Our complete Hopf equation "knows" about this exponent inequality via Eq. (7) since by the same majorization, one can show the inertial range exponents of $\langle \epsilon_1 \theta_3 \theta_4 \rangle$ are positive; however, this has not yet emerged directly by solving the full eigenvalue problem.

For the three point function, the 0th order term in our Hopf equation, L_0, gave exponents similar to experiment and a derivative skewness Reynolds' independent. For small L_D, higher normalized moments of $\partial\theta$ diverge with R. Thus the large scale gradient $\vec{g}$ is "relevant" to the small scales in the same way as are fluctuations in ϵ_θ. Our phenomenological decomposition of L_H into L_0 plus a nonintegrable, nonuniversal term L_D lead to a similar decomposition of ψ_3, i.e., (8) (details of the crossover calculation for ψ_4 suggest the analogous relation is more L_D dependent), and facilitated solution of the singular perturbation problem posed by L_D, to which we could only allude here [14]. The white noise Hopf equation expanded around $\zeta = 0$ has been solved in this way [14].

In contrast to most problems, our symmetry SL(n) is a property of the $(n+1)$-point correlation function only and not of the θ field being averaged in the correlation function. One may hope it is a more robust property of the dynamics than the scaling exponents and therefore visible at lower Reynolds' number, but an experiment to check (8) is required to quantify the effects of L_D.

It is not surprising that the intermittency exponents of passive scalar mixing are no more universal than the velocity field itself, even if Gaussian [15] (but note $\langle \theta_1 \theta_2 \rangle$ satisfies an inhomogeneous equation and thus inherits the velocity exponent). This may seem a particularity of passive scalars, if one believes the small scales of turbulence universal; we rather believe that the scalar analysis shows how the inertial range velocity can be directly imprinted by large scale anisotropy leading to nonuniversal exponents for correlators as low as 3rd order [16].

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[11] References [7,8] assume the velocity is white and therefore work from (2). Further approximations, either large d but ($\zeta = \frac{2}{3}$) or expansion around $\zeta = 2$ and general d, are required to do perturbation theory.

[12] In order to match the inertial range Hopf solutions to those in the dissipation range all the zero modes are required. In particular, those with negative $\lambda = -|\lambda_d|$ then appear in the inertial range as correction terms but capable of influencing inertial range experiments if $|\lambda_d| \ll 1$.

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