Finite-Time Singularities in the Axisymmetric Three-Dimension Euler Equations

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For pointlike singularities localized well away from the symmetry axis, axisymmetric flows with swirl are arbitrarily well approximated by two-dimensional Boussinesq convection. An adaptive mesh simulation of the latter equations was continued until the maximum three-dimensional vorticity showed a factor of 10^7 increase, allowing a reasonable determination of exponents, and elucidation of the mechanism of blowup.

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Understanding whether smooth initial conditions to the three-dimensional Euler equations develop singularities (infinite velocity derivatives) in a finite time is an important step in understanding high-Reynolds-number hydrodynamics. Focusing on singularities is a sharp way of asking how excitation is passed onto small scales, a process called "vortex stretching" in the turbulence literature. Finite-time blowup is essentially the condition that the equations act nonlinearly. Exponential growth is good evidence that the strain (essentially the derivative of logarithm of vorticity) is coming from modes and a region of space remote from the vorticity in question. Finally, because plausible singularities are spatially localized, adaptive mesh codes can be used to follow the collapse to very small scales. The computational problem is much simpler than for steady-state turbulence, where the scaling regime cannot be attained with existing computers.

The simplest dimensional treatment of the Euler equations predicts a $1/(t^*-t)$ divergence for velocity derivatives. Current numerical simulations, however, show a crossover to exponential growth with the most intense vorticity organized into sheets, passively strained [1,2]. Whether the initial conditions were anomalous, the range of scales inadequate, or the singularities "unstable" (i.e., think of rolling a marble into a volcano) is not known.

Grauer and Sideris [3] observed that axisymmetric flows with swirl [the velocity is three dimensional but independent of ϕ in cylindrical coordinates (r,ϕ,z)] do not exclude finite-time blowup, yet computationally look two dimensional. For flows confined radially to a thin shell, the 3D axisymmetric Euler equations are virtually equivalent to the two-dimensional Boussinesq equations without dissipation. It is these equations that we simulate and in which we find a singularity. Second, the convection analogy dictates that the vorticity in axisymmetric flows organizes into sheets (i.e., thermal plumes), and that our singularity which occurs on the plume tip is generic. The maximum 3D vorticity diverges as $(t^*-t)^{-2}$.

The axisymmetric flow equations are written

$$\partial_t(rv_\phi)+\frac{1}{r}\frac{\partial(\psi,rv_\phi)}{\partial(r,z)}=0\,, \tag{1a}$$

$$\partial_t(\omega_\phi/r)+\frac{1}{r}\frac{\partial(\psi,\omega_\phi/r)}{\partial(r,z)}=\frac{-1}{r^4}\partial_z(rv_\phi)^2\,, \tag{1b}$$

$$r\partial_r\left(\frac{1}{r}\partial_r\psi\right)+\partial_z^2\psi=-r\omega_\phi\,. \tag{1c}$$

For a solution centered on $r\sim r_0$, we find, under the substitutions $\frac{1}{2}r^2\to\frac{1}{2}r_0^2-y$, $z\to x/r_0$, $t\to t/r_0$, $(rv_\phi)^2\to\theta$, $\omega_\phi/r\to-\omega$,

$$\partial_t\theta+\frac{\partial(\psi,\theta)}{\partial(x,y)}=0\,, \tag{2a}$$

$$\partial_t\omega+\frac{\partial(\psi,\omega)}{\partial(x,y)}=\frac{r_0^{-4}}{(1-2y/r_0^2)^2}\partial_x\theta\,, \tag{2b}$$

$$\partial_x^2\psi+(1-2y/r_0^2)\partial_y^2\psi=(1-2y/r_0^2)\omega\,. \tag{2c}$$

Hence for $|y|\ll r_0^2$ we obtain the Boussinesq equations [i.e., drop $2y/r_0^2$ in comparison to 1 in (2b) and (2c)], with nondimensionalized gravitational acceleration r_0^{-4}. The perturbation represented by y/r_0^2 is regular and diminishes in importance near the singularity time where the relevant range of y is $O((t^*-t)^2)$. For future reference the three-dimensional vorticity $(\omega_r,\omega_\phi,\omega_z)$ scales as $(\partial_x\theta^{1/2},\omega,\partial_y\theta^{1/2})$.

The Boussinesq equations were simulated by mapping the plane $-\infty<x,y<+\infty$ onto the unit square $0\le u,v\le 1$ by $x=[a_x+b_x\cos(\pi u)+c_x\cos(2\pi u)]\cot(\pi u)$ (similarly for y,v), with the constants adjusted to maintain resolution. The equations were finite differenced on a 256^2 uniform mesh in (u,v) with the Osher-Chakravarty algorithm [4], the Poisson equation was inverted by cyclic reduction [5], and a Runge-Kutta algorithm with step-size control was used for time advancement. The incipient singularity was kept near $u,v\sim\frac{1}{2}$, where the resolution was highest, by a time-dependent spatially uniform wind.

Since our singularity had comparable dimensions in x and y, and the strain was generated locally, the task of coordinate adjustment was greatly simplified. When we had fewer than 12–18 grid points across the plume tip, we stopped the integration and interpolated a mesh with new constants a_i,b_i,c_i. In the process, some vorticity was pushed out to within a few mesh points of $u,v=0,1$, where it was truncated to preserve the boundary conditions. The odd and even meshes were averaged to eliminate a well-known and weak instability of centered differences.

Statistics were computed before and after the coordinate adjustment. We can therefore say with confidence that resolution errors in max(ω) and max($|\nabla\theta|$) were held to $\lesssim 6\%$ at all times. The loss in strain due to the elimination of vorticity around "infinity" averaged $\sim 10\%$ per factor of 2 growth in $|\nabla\theta|$. The ratio of the inner scale, min($|\nabla\theta|^{-1}$), to those affected by the cutoff at "infinity," $\sim$50–100, is then a quantitative measure of the locality of the strain. Since our differencing scheme adds diffusion locally, proportional to resolution, our errors tend to inhibit the singularity. In addition, we redid a portion of our run with 384^2 resolution and examined the convergence from coarser 128^2 and 192^2 meshes. We again found that per factor of 2 growth in $|\nabla\theta|$, finite resolution led to a $\sim 7\%$ underestimate. Further details may be found in Ref. [6].

It is both technically convenient and conceptually neat, without in any way contravening the above error estimates, to continuously adjust scales. First note for arbitrary time-independent l,η that the Boussinesq equations are invariant under $[\mathbf{r}=(x,y)]$

$$\begin{aligned}\theta &\to \text{const}+l^{\eta}\theta(\mathbf{r}/l^{2+\eta},t/l)\,,\\ \omega &\to l^{-1}\omega(\mathbf{r}/l^{2+\eta},t/l)\,.\end{aligned} \tag{3}$$

(The scale factors on r,t,θ preserve the effective gravitational acceleration.) Now imagine l depends on t, $\alpha(t)\equiv -\partial_t l$, and $\partial_t T = l^{-1}$ defines T. Then the Boussinesq equations can be written for new dependent variables $\Theta = l^{-\eta}(\theta-\text{const})$ and $\Omega = l\omega$ as functions of $\mathbf{R}\equiv \mathbf{r}/l^{2+\eta}$ and T, with $\alpha(T)$ and η as parameters as yet undetermined.

In analogy with Ref. [7], a finite-time singularity with $l\sim(t^*-t)$ and $T\sim -\ln(t^*-t)$ exists if it is possible to choose $\eta\geq 0$ and $\text{const}_1 \geq \alpha(T) \geq \text{const}_2 > 0$ such that for $|R|\sim 1$ and all T the total variation in Θ, Ω, and their gradients is of order 1. It is not obvious, and is just a restatement of our principal result, that all these conditions can be met by adjusting only two parameters. Physically, η will turn out to be quite small and is used to make the change in θ across the region of large gradient of order 1; α or l maintain the length scale of the singular region at order 1. Since the precise numerical value of this scale is immaterial, there is some arbitrariness in the instantaneous value of α, but its average is well defined.

There is no reason for the dependence of Ω and Θ on T to disappear and indeed it does not; our singularity continuously evolves in shape. Note that in the R,T variables there is nothing large or small and once $\alpha(T)$ is selected, the numerical problem is equivalent to simulating any other partial differential equation with a finite, and not terribly large, number of degrees of freedom. Both discrete and continuous methods for adjusting coordinates were implemented and agreed well.

Figure 1 gives an overview of our data and shows how the maximum *three-dimensional* vorticity diverges [N.B., $\omega_{r,z}\sim|\nabla\theta|\gg\omega_\phi$ by Eq. (3)]. The initial condition was $\theta=(1+0.2y)/(1+x^2+y^2)^2$, $\omega=0$. For $t\lesssim 8.5$, the temperature evolves into a thermal plume of lateral extent $\delta x\sim 1$ with a sharp gradient in θ along its leading edge which is otherwise smooth. The strain which corresponds to stagnation flow around the tip for $x=0$ stabilizes the bubble cap and is roughly constant, so $|\nabla\theta|$ grows exponentially. The cap is always unstable for wavelengths greater than or equal to the thickness σ, but the strain suppresses the instabilities' amplitude, increases their wavelength, and advects them away from the center. Finally, when the ratio of σ to the radius of curvature $r_c, \sigma/r_c\sim 10^{-2}$, time dependence, which is always present due to the edges of the plume and the increasing vortex sheet strength, leads to large Kelvin-Helmholtz instabilities on the side of the cap and Rayleigh-Taylor-like instabilities near the $x=0$ symmetry axis (Fig. 2). We have verified with a linearized analytic calculation that for the background strain, tip curvature, and cap thickness we calculate for $t\sim 8$, instabilities are predicted to grow out of a linear regime before they are advected off the cap [8,9].

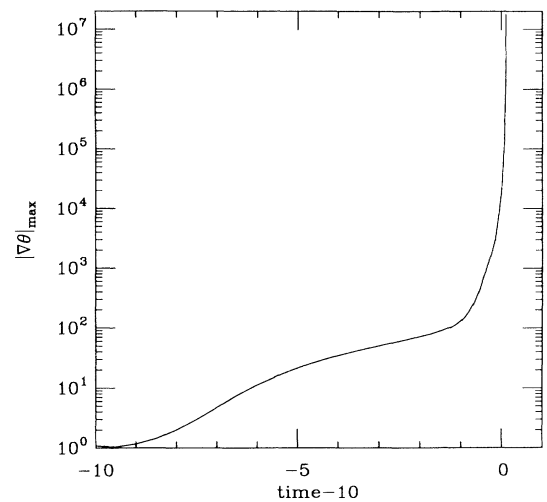

FIG. 1. The maximum temperature gradient (or 3D vorticity) as a function of time for Eq. (2).

The true blowup begins at $t\sim 8.5$. A smooth cap never reforms, the point of maximum $|\nabla\theta|$ is always on or near the symmetry line $x=0$, and the radius of curvature of the iso-θ contour near the singularity is larger than but of the order of the thickness (Fig. 3). Analytic estimates [6] suggest that the rollup is not singular. (The mechanism by which a cusp forms on a vortex sheet [10] is not relevant here.) The shape of the singular region is always changing, with new instabilities being born and pushed towards infinity in the rescaled coordinates as the code maintains resolution around the incipient singularity. Since the singularity develops so rapidly, the large, outer scales are effectively frozen. A series of pictures analogous to Fig. 3 for $t\geq 9$ would naturally telescope. At the

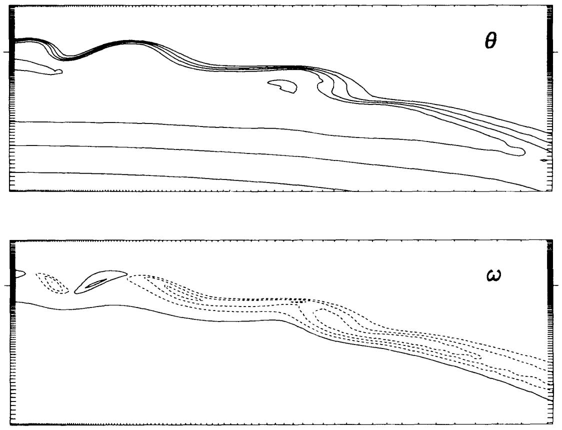

FIG. 2. Contour plots of θ and ω from Eq. (2), $t=8.8$, showing the bubble cap destabilizing. The bubble is moving upwards, $\theta=0.2$–1.0 in units of 0.2. The grid points are shown by ticks, $0\le x\le 0.8$, and the y range is 0.33. Note the "wrong"-signed vorticity near $x=0$, indicative of a Rayleigh-Taylor "finger," and the rollups for larger x.

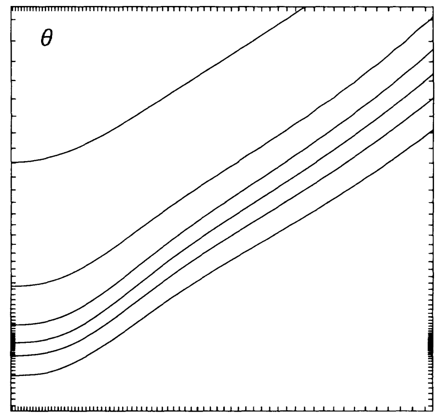

FIG. 3. Contour plot of θ at $t=10.077$ with $\max(|\nabla\theta|)=1.9\times10^{5}$ and located on the $x=0$ symmetry line, showing that the singularity is well resolved. The coordinate range is $0<x<8.6\times10^{-5}$ with y magnified 5×, and θ varies from 0.9 to 0.4 as y increases. The cutoff near "infinity" begins to act on θ for $|x|\ge 1.4\times10^{-3}$.

last time $\max(|\nabla\theta|)\sim 1.8\times10^{7}$, the innermost mesh spacing is $\sim 2\times10^{-9}$, and the outer scale beyond which the vorticity is truncated (i.e., infinity) is $\sim 10^{-5}$. There was no impediment to further integration.

To infer exponents we replot our data as in Fig. 4 to bring out the expected linear behavior [cf. Eq. (3) and note that the strain should scale as ω]. There are no fitting parameters, and the ripple in the data is real and due to shape changes. The diverging strain, which equals $\partial_t \ln|\nabla\theta|^2$, is conclusive evidence that the growth in the three-dimensional vorticity is not exponential. We can also determine $\eta=0.2\pm0.1$ approximately from the total variation in θ across the region of large gradients. We therefore have, in conformity with (3) near the singularity (v = velocity),

$$\theta=\text{const}+\tau^{\eta}\Theta(\mathbf{r}/\tau^{2+\eta},T)\,,$$
$$\mathbf{v}=\text{const}+\tau^{1+\eta}\mathbf{V}(\mathbf{r}/\tau^{2+\eta},T)\,, \qquad (4)$$

where $\tau=(t^*-t)$ and $T\sim-\ln(\tau)$. (Note that for the inviscid Burgers equation the singular part of the velocity also vanishes at $\tau=0$ although its gradient is infinite.)

In summary, a singularity develops in the center of the leading edge of a thermal plume, through formation of a cusp by a Rayleigh-Taylor-like mechanism. Analytic estimates [6] show that the radius of curvature and thickness must go to zero together if there is to be a singularity, as indeed we found. The local stretching (or elongation of constant θ contours) diverges as $1/\sigma$ by area preservation.

Several salient conclusions for axisymmetric Euler flow should be noted. The dynamics are local in (r,z); the singularity is pointlike, and its shape is unsteady. The r,z components of the three-dimensional vorticity diverge as τ^{-2}, while $\omega_\phi=\omega\sim\tau^{-1}$. Taking account of the length scaling in (4) we find $\int|\omega_{3D}|^{2+a}d^3x$ diverges for $a>\eta$. The largest velocity gradient is $\partial_r v_\phi$ and we obtain the "conventional" vorticity strain relations [11,12], namely, the vorticity parallel to the eigenvector of the intermediate (small) eigenvalue of the rate of strain matrix.

Although we are only able to track one and presumably

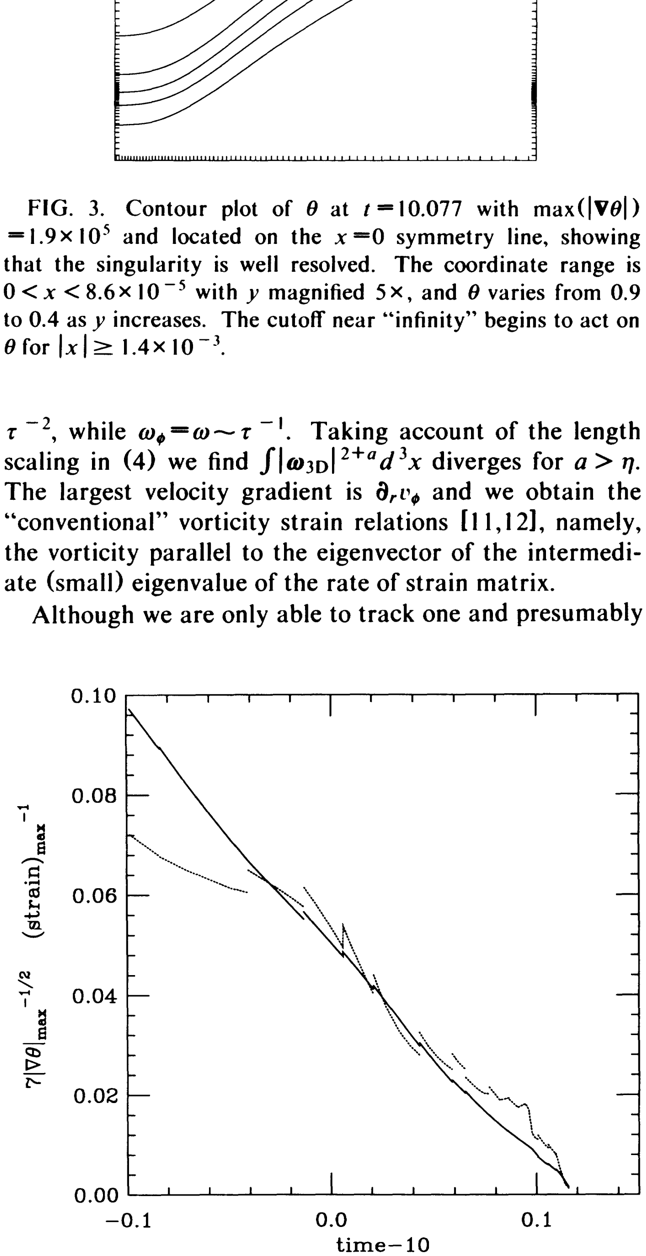

FIG. 4. The inverse strain (dashed) and $|\nabla\theta|$ from Fig. 1 plotted to appear linear [cf. Eq. (4)]. The steps show when the coordinates were adjusted and quantify the resolution errors. The ripple is real and can be correlated with shape changes in the solution. In particular, the bump in the strain for $t\sim0.095$ occurs because the location of the maximum moves to a pair of points off the symmetry axis.

the first singularity, other points on the bubble cap should eventually blow up similarly. Viscosity ν will terminate the blowup [13] on a small scale $\sim\nu^{(2+\eta)/(3+2\eta)}$. We suspect that perturbations breaking the axisymmetry will grow, leading to flows qualitatively like the paired vortex tubes studied in Ref. [2]. While we now have good numerical evidence that singular solutions to the 3D Euler equations exist, it is of great interest to understand why singularities have proved so elusive for nonsymmetric initial conditions.

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