Invariants for the one-point vorticity and strain rate correlation functions

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An algorithm to enumerate the number of independent scalars that determine the general tensor formed from n velocity gradients at a point in homogeneous-isotropic turbulence is elaborated for $n = 4$. The physical content of the invariants that result as well as their determination from experiment is discussed.

Tensors formed from n velocity derivatives evaluated at a common point ($\partial_a = \partial/\partial x_a$),

$$T^n_{a,b,\ldots} = \langle(\partial_a v_b)(\partial_c v_d)\ldots\rangle ,$$

have proved to be a convenient characterization of the small-scale structure of fully turbulent flows.[1] Selected elements are easily measured with hot wires while more general combinations of terms, such as the correlations between the vorticity and rate of strain, have an immediate physical interpretation. When the small scales are homogeneous and isotropic, symmetry arguments alone can provide constraints among measured quantities as well as relate them to correlations of greater physical interest.[2,3] Thus, $-\langle(\partial u_1/\partial x_1)^3\rangle$ is proportional to both the fourth moment of the energy spectrum and the rate of production of vorticity by stretching, $\langle\omega_a e_{ab}\omega_b\rangle$, where $e_{ab} = \frac{1}{2}(\partial_a v_b + \partial_b v_a)$ and $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. Repeated indices are summed from 1 to 3 and tr denotes the trace of a matrix.

The most common measure of small-scale intermittency in fully developed turbulence is the flatness factor of the longitudinal velocity derivative, $\langle(\partial u_1/\partial x_1)^4\rangle / \langle(\partial u_1/\partial x_1)^2\rangle^2$, and is thus related to one component of T^4.[1] Just as the transfer of energy via vortex stretching is most naturally and physically expressed in terms of the correlation between vorticity and strain rather than the skewness, it is also fruitful to describe intermittency in the same way. The four quantities $\langle[\mathrm{tr}(e^2)]^2\rangle$, $\langle\mathrm{tr}(e^2)\omega^2\rangle$, $\langle\omega_a e_{ab} e_{bc}\omega_c\rangle$, and $\langle\omega^2\omega^2\rangle$ are all expressible in terms of T^4, and thereby generalize the conventional flatness in a manner so as to express in a rotationally invariant way the correlations between vorticity and strain. How they vary with R_λ, and the extent to which they deviate from their Gaussian values, would indicate whether both the vorticity and strain rate were intermittent, and if so, whether bursts in one were correlated with bursts in the other. Further study could also yield important information on the velocity field in the vicinity of the active regions.

Of course, it would be virtually impossible to simultaneously measure all eight components of $\partial_a v_b$ and fortunately, also unnecessary. The assumption of statistical isotropy and homogeneity can be used to relate the four vorticity-strain correlations defined above to an equal number of other elements of T^4 that could be measured by currently available techniques. The numerical relations between these two sets of correlations constitute the second-half of our paper and we conclude by considering how knowledge of additional elements of T^4 could serve to sharpen the phenomenological theories of intermittency that now exist.

We begin by enumerating the number of invariants that under the assumption of isotropy (proper rotations and reflections) suffice to completely specify T^4. We will then show that homogeneity imposes no additional constraints among these invariants, and therefore, unless some nontrivial use is made of the Navier–Stokes equations, four independent numbers are required to specify T^4. Recall that under the same assumptions only one invariant is required to completely determine T^3.[3,4] Although a field that is isotropic at every point is necessarily homogeneous, if we consider only the average of velocity derivatives at a single point the two symmetries impose distinct restrictions on the elements of T^n.

Isotropy implies that T^n can be expressed as the sum of $(2n)!/(2^n n!)$ terms (i.e., the number of ways of partitioning $2n$ indices into pairs), each the product of n factors of the unit matrix δ_{ab}. (The product of two of the completely antisymmetric Levi–Civita symbols ϵ_{abc} can be reexpressed as the sum of products of the unit tensor.) The number of invariants as well as the expression of an arbitrary tensor element in terms of the invariants can then be found by imposing permutation symmetry and incompressibility on the above sum.

The arithmetic required to implement this algorithm becomes prohibitive for $n \geq 4$ and is unnecessary once it is realized that the number of independent coefficients remaining after all symmetries are imposed equals the number of distinct scalars that can be formed by contracting the tensor indices. This enumeration is rendered trivial if we write

$$\partial_a v_b = e_{ab} + \tfrac{1}{2}\epsilon_{abc}\omega_c . \tag{1}$$

In addition, we need the identity for small x

$$\det(1 + xe) = \exp\{\mathrm{tr}[\ln(1 + xe)]\}, \tag{2}$$

where "tr" and "det" stand for trace and determinant, respectively. A Taylor expansion in x then yields for $\mathrm{tr}(e) = 0$.[3]

$$\mathrm{tr}(e^3) = 3\det(e), \quad [\mathrm{tr}(e^2)]^2 = 2\,\mathrm{tr}(e^4) . \tag{3}$$

In general, the trace of e_{ab} to any power can be reduced to sums of products of $\det(e)$ and $\mathrm{tr}(e^2)$. The scalars that can be formed from T^n can be classified, after substituting (1), by the number of factors of $\boldsymbol{\omega}$, which is necessarily even. (Otherwise, the corresponding

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tensor would have an odd number of indices and thus contain a factor of ϵ_{abc} in its expansion. But ϵ_{abc} is odd under reflection while ω and e_{ab} are both even. For $n<6$ odd terms in ω can be ruled out without invoking reflection symmetry.)

For $n=4$ there are precisely four invariants:

$$I_1=\langle(\mathrm{tr}e^2)^2\rangle,\quad I_2=\langle\omega^2\mathrm{tr}e^2\rangle\,,$$
$$I_3=\langle\omega_a e_{ab}e_{bc}\omega_c\rangle,\quad I_4=\langle(\omega^2)^2\rangle\,. \tag{4}$$

There are thus 101 constraints among the 105 terms in the expansion of T^4. Were there any fewer, it would imply the existence of a scalar term in addition to the above, while by inspection none exist. If there were any more than 101 constraints, then several of the I_α would be linearly related, but rotations cannot turn a pure strain into a rotation or vice versa.

The number of invariants of order n involving only e is simply the number of distinct ways of partitioning n into the sum of n_2 factors of 2 and n_3 factors of 3. For reasonable n the invariants involving ω can be enumerated by inspection. Thus for T^n, there are a total of 5 invariants for $n=5$ and 10 for $n=6$.

For $n=3$ our enumeration yields two invariants under rotations $\langle\mathrm{tr}e^3\rangle$ and $\langle\omega_a e_{ab}\omega_b\rangle$, respectively. Homogeneity then implies $\langle\partial_a v_b\partial_b v_c\partial_c v_a\rangle=0$ or $\langle\mathrm{tr}e^3\rangle=-3\langle\omega_a e_{ab}\omega_b\rangle/4$.[2,3] For the fourth-order invariants (4) (and by suggestion for $n>4$), however, homogeneity implies no further restrictions beyond those already imposed by isotropy.

For $n=4$ we prove this assertion by supposing that there is a condition, $\sum_{\alpha=1}^4 c_\alpha I_\alpha=0$, with c_α constant, and then constructing a series of ensembles that are manifestly homogeneous and isotropic. Since the c_α are by assumption independent of the ensemble, we will be able to show $c_\alpha=0$. Our test ensembles are constructed from a two-dimensional velocity field with variation only in the plane in which the velocity lies. An angular average is performed to achieve isotropy.

In any such ensemble $I_3=0$. Let us further specialize to $v_1=\sin(k_a x_2)$, $v_2=\sin(k_b x_1)$ and compute $s=\mathrm{tr}e^2+\frac{1}{2}\omega^2$ and $d=\mathrm{tr}e^2-\frac{1}{2}\omega^2$. A spatial average is then required to achieve homogeneity. It is then easily seen that while $\langle\mathrm{tr}e^2\rangle=\frac{1}{2}\langle\omega^2\rangle$, $\langle sd\rangle=0$ and $\langle s^2\rangle/\langle d^2\rangle$ depends on k_a/k_b and therefore on the ensemble. Thus, homogeneity does not allow a linear relationship of the assumed form among I_1, I_2, and I_4, i.e., $c_1=c_2=c_4=0$. Finally, taking an arbitrary three-dimensional velocity field and still assuming $\sum_{\alpha=1}^4 c_\alpha I_\alpha=0$, implies $c_3=0$ and completes the demonstration that the I_α are linearly independent.

Although our enumeration of the invariants of the general tension T^4 was considerably easier than expanding it in terms of δ_{ab}, it remains to relate an arbitrary element of T^4 to the I_α. We again proceed by resolving the velocity gradients into strain and vorticity.

A tensor consisting of a string of elements from e_{ab} must be proportional to I_1. The constant of proportionality can be determined by evaluating all quantities in a Gaussian ensemble. The required averages are most conveniently evaluated by differentiating the generating function

$$\begin{aligned}F(\lambda,\mu)&=\ln\Big[\int(\Pi_{a\geq b}de_{ab})\delta(\mathrm{tr}e)\exp\Big(-\sum_a\lambda_a e_{aa}^2\\&\qquad-2(\mu_1e_{23}^2+\mu_2e_{13}^2+\mu_3e_{12}^2)\Big)\Big]\\&=-\tfrac{1}{2}\ln(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3)-\tfrac{1}{2}\ln(\mu_1\mu_2\mu_3)\end{aligned}\tag{5}$$

with respect to λ_a or μ_b and setting $\lambda_a=\mu_b=\lambda$ at the end to achieve isotropy. An additive constant has been omitted in the second line of (5). One then finds

$$\langle e_{11}^4\rangle=4I_1/105,\quad\langle e_{12}^4\rangle=3I_1/140,$$
$$\langle e_{11}^2e_{12}^2\rangle=I_1/105,\quad\langle e_{12}^2e_{13}^2\rangle=I_1/140\ldots\,. \tag{6}$$

An arbitrary element of the tensor formed from two factors of vorticity and two factors from e_{ab} may be expressed as a linear combination of I_2 and I_3. Since the general tensor has only six indices, it may be written explicitly as

$$\begin{aligned}\langle\omega_a\omega_b e_{cd}e_{fg}\rangle&=\alpha\delta_{ab}\delta_{cd}\delta_{fg}-\tfrac{3}{2}\alpha\delta_{ab}(\delta_{cf}\delta_{dg}+\delta_{cg}\delta_{df})\\&+\beta(\delta_{ac}\delta_{bf}\delta_{dg}+\delta_{ac}\delta_{bg}\delta_{df}+\delta_{ad}\delta_{bf}\delta_{cg}+\delta_{ad}\delta_{bg}\delta_{cf}\\&+\delta_{af}\delta_{bc}\delta_{dg}+\delta_{af}\delta_{bd}\delta_{cg}+\delta_{ag}\delta_{bc}\delta_{df}+\delta_{ag}\delta_{bd}\delta_{cf})\\&+\tfrac{4}{3}\beta(\delta_{ab}\delta_{cf}\delta_{dg}+\delta_{ab}\delta_{cg}\delta_{df}-\delta_{ac}\delta_{bd}\delta_{fg}-\delta_{ad}\delta_{bc}\delta_{fg}\\&-\delta_{af}\delta_{bg}\delta_{cd}-\delta_{ag}\delta_{bf}\delta_{cd}),\end{aligned}\tag{7a}$$

where we have defined

$$\alpha=8I_3/105-I_2/21,\quad\beta=3I_3/70-I_2/70\,. \tag{7b}$$

Equation (7a) was derived by imposing the permutation symmetries and $\mathrm{tr}e=0$ on the 15 terms that describe a six-index tensor under isotropic conditions. Finally, for the tensor involving only the vorticity we have

$$\langle\omega_a\omega_b\omega_c\omega_d\rangle=(I_4/15)(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\,. \tag{8}$$

Either additional assumptions or some further input from the Navier–Stokes equations is required to say anything more about the I_α. Lower bounds on I_α can be given in terms of $\langle\omega_a e_{ab}\omega_b\rangle=-35\langle(\partial u_1/\partial x_1)^3\rangle/2$. One finds

$$\begin{aligned}&\langle\omega_a e_{ab}\omega_b\rangle^2\leq I_3\langle\omega^2\rangle\,,\\&\langle\omega_a e_{ab}\omega_b\rangle^2\leq 2I_4\langle\mathrm{tr}e^2\rangle/5\,,\\&\langle\omega_a e_{ab}\omega_b\rangle^2\leq 2I_2\langle\omega^2\rangle/3\,.\end{aligned}\tag{9}$$

By working from $\langle\mathrm{tr}e^3\rangle$, Betchov derived an analogous bound for I_1.[3] One can also show

$$I_3\leq 2I_2/3,\quad I_3^2\leq 3I_1I_4/10,\quad I_2^2\leq I_1I_4\,. \tag{10}$$

The inequalities (9)–(10) are optimal if nothing more than $\langle\omega^2\rangle=2\langle\mathrm{tr}e^2\rangle$, isotropy, and incompressibility is assumed. The configuration for which equality holds in (9), $e_{aa}\propto(2,-1,-1)$, $e_{a\neq b}=0$, and $\omega\propto(1,0,0)$, does not satisfy $0<-\det(e)=\omega_a e_{ab}\omega_b/4$ so the inequalities could be somewhat strengthened if homogeneity were imposed. The bounds on I_3 in (10) might be improved if a value of the skewness is imposed externally on the ensemble.

The only other reliable information we have on the I_α for $\alpha>1$ comes from fully resolved numerical simulations for R_λ in the range 60–90.[5] In Table I we reproduce the data of Ref. 5 with a different normalization together with the values of I_α appropriate to a Gaussian random velocity field. Since the skewness is of order 0.5, i.e., $\langle \omega_a e_{ab} \omega_b \rangle^2 = 0.18 \langle \mathrm{tr} e^2 \rangle^3$, none of the inequalities in (9)–(10) is particularly stringent.

Experiments done with crossed wires yield three independent measures of the fourth-order velocity gradient statistics

$$F_1 = \langle (\partial u_1/\partial x_1)^4 \rangle, \quad F_2 = \langle (\partial u_1/\partial x_1)^2 (\partial u_2/\partial x_1)^2 \rangle,$$
$$F_3 = \langle (\partial u_2/\partial x_1)^4 \rangle. \tag{11}$$

The fourth quantity one might hope to obtain with wires, $\langle (\partial u_2/\partial x_1)^2 (\partial u_3/\partial x_1)^2 \rangle$, equals $F_3/3$ as may be seen by writing $2F_3 = \langle (\partial \tilde{u}_2/\partial x_1)^4 \rangle + \langle (\partial \tilde{u}_3/\partial x_1)^4 \rangle$ with $\tilde{u}_{2,3} = (u_2 \pm u_3)/\sqrt{2}$ and rearranging. After re-expressing $\partial u_a/\partial x_1$ in terms of strain and vorticity, Eqs. (6)–(8) imply

$$F_1 = 4I_1/105,$$
$$F_2 = I_1/105 + I_2/70 - I_3/105,$$
$$F_3 = 3I_1/140 + 11I_2/140 - 3I_3/35 + I_4/80. \tag{12}$$

Clearly, one additional experimental number is needed to fully determine T^4. It might either be an independent determination of one component of the vorticity or

$$F_4 = \langle (\partial u_1/\partial x_1)^2 (\partial u_2/\partial x_3)^2 \rangle$$
$$= I_1/105 + I_2/210 + 2I_3/105.$$

But, note that $\langle (\partial u_1/\partial x_1)^2 (\partial u_1/\partial x_2)^2 \rangle = F_2$ and $\langle (\partial u_3/\partial x_1)^2 (\partial u_3/\partial x_2)^2 \rangle = F_3/3$.

We believe that more will be learned about small-scale intermittency by determining I_α in a high-quality wind tunnel than by measuring just F_1 at ever higher R_λ. The case for laboratory experiments becomes stronger if second derivatives can be resolved. All phenomenological theories of intermittency to date have parametrized the fluctuations with a single-scalar field interpreted variously as the energy transfer or dissipation.[1,6] There is no freedom for different velocity derivatives to scale differently with Reynolds number, i.e., I_α/I_1 must approach a constant as R_λ increases.

A rather different conclusion is suggested by the alternative viewpoint that intermittency arises from an assembly of identifiable and persistent vortical structures that become more singular as R_λ increases.[7,8] (Cascade notions play little or no role in this picture.) For the solutions we have examined,[7] in which the vortex tube or sheet is maintained by a smooth background straining field, $e_{ab}\omega_b$ is always linear in the intermittent part of the velocity field. One would therefore predict that I_3/I_1 tends to zero with increasing R_λ. A comparison between the computed and Gaussian values of I_α in Table I suggests a similar conclusion. Our argument is, of course, only heuristic and ultimately leads to conflict with the first inequality in (9), if one assumes that the skewness scales as a positive power of R_λ. It does, however, emphasize that the invariants I_α provide important information on the possible flow configurations in the intermittent regions.

TABLE I. The invariants in Eq. (4) all normalized by $\langle \mathrm{tr} e^2 \rangle^2$. The numerical values are taken from run (2b) of Ref. 5 with $R_\lambda \sim 60$–90. Only two figures are significant.

	I_1	I_2	I_3	I_4
Gaussian	7/5	2	2/3	20/3
Numerical	2.58	4.85	0.733	15.6

Lastly, we recall the long-standing question of the nature of the singularity predicted by the inviscid Navier–Stokes equations when initialized with a velocity restricted to low wavenumbers and run forward in time.[9] If one could be assured that the small scales were isotropic, perhaps by using random initial conditions and averaging, then the question would arise whether all the I_α diverged in the same way. The various moments of the energy spectrum may not be the most illuminating way to characterize this singularity.

Orszag has enumerated the number of scalars necessary to specify the general n velocity correlation function in wavenumber space.[10] We have not found that classification particularly useful here since homogeneity did not provide any further constraints among the I_α.

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