Incipient Singularities in the Navier-Stokes Equations

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Infinite pointwise stretching in a finite time for general initial conditions is found in a simulation of the Biot-Savart equation for a slender vortex tube in three dimensions. Viscosity is ineffective in limiting the divergence in the vorticity as long as it remains concentrated in tubes. Stability has not been shown.

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The incompressible Navier-Stokes equations are used in many practical problems, yet it has never been proven that their solutions remain finite for all times even if there are no external forces.[1-3] In this Letter we suggest physical reasons why a proof may be difficult to achieve. Within our model, vortex tubes generically collapse in a finite time in such a way that the viscosity only becomes important when the velocity and its derivative are formally of the order of the exponential of a Reynolds number. Other processes must intervene earlier and the most likely candidate is an inviscid deformation of the tubes into ribbons where the viscosity will then control the stretching. If our solutions are a stable up to this point then they are a very efficient means of transferring and focusing energy into small scales. They may partially explain the violent intermittency seen in wall-bounded shear flows.

Our construction proceeds from the well-known equivalence between the velocity field from a slender vortex tube and the Biot-Savart formula with a suitable cutoff.[4] Under time evolution, the connection with slender tubes is never lost, i.e., the filament radius of curvature is always appreciably greater than the local core size. A local approximation to the Biot-Savart formula is then obtained which confirms certain numerical results analytically. Solutions to the Navier-Stokes or Euler equations may in principle be recovered by perturbation methods which exploit the slow variation along the filament axis. The extent to which the cores deform may be estimated by similar means and is also examined numerically. Our emphasis here will be on existence.

Rigorous results severely restrict any possible singularities that solutions to the Navier-Stokes equations can assume. Leray[1] showed in effect that either the velocity is everywhere smooth or its maximum magnitude becomes unbounded according to

$$\max(v) \geq \mathrm{const}[\nu(t^* - t)]^{1/2}, \qquad (1)$$

where ν is the kinematic viscosity, t^* the singularity time, and const an arbitrary constant. The analogous lower bound on the maximum vorticity, ω, is const$/(t^* - t)$ (neglecting technicalities).[3] Scheffer was the first to bound the space-time dimension of the singular set and the best current estimates place the Hausdorff dimension strictly less than 1.[2,3] A different line of research has proven existence for the Laplacian to a power greater than 1.[5]

Though arguments originating from Kolmogorov's theory of inertial-range turbulence suggest that the Euler equation should have a finite-time singularity, numerical experiments based on a Fourier representation have proved inconclusive.[6] Chorin did find a singularity with vortex methods, but his solution looks very different from ours and there are worries about the numerical procedures.[7]

The Biot-Savart formula with a core size σ reads[4]

$$\mathbf{v}(\theta) = \frac{-\Gamma}{4\pi} \int \frac{[\mathbf{r}(\theta) - \mathbf{r}(\theta')] \times (d\mathbf{r}/d\theta')d\theta'}{\{[\mathbf{r}(\theta) - \mathbf{r}(\theta')]^2 + \sigma^2(\theta) + \sigma^2(\theta')\}^{3/2}}, \qquad (2a)$$

$$\sigma^2\, ds/d\theta = \mathrm{const}, \qquad (2b)$$

where θ is a Lagrangean parameter, s is the arc length, and Γ is the circulation. If several filaments are present, each convects all the others. Equation (2b) ties the core size to the local stretching. The maximum velocity and vorticity scale as Γ/σ and Γ/σ^2, respectively. Infinite stretching ($\sigma \to 0$) in a finite time is equivalent to blowup. The reduction of the velocity field of a vortex tube to (2a) is rigorous[4] while (2b) is only one extreme of a continuum of models that preserve volume.

To solve (2) numerically we lay down a series of nodes, fit the curve $r(\theta)$ with cubic splines, and solve (2b) for $\sigma^2(\theta)$. The integral over θ' was done with Simpson's rule to yield the velocity at each node θ. The nodes are then stepped forward in time by use of a variable—step-length Runge-Kutta-Fehlberg algorithm. Questions of resolution, accuracy, and stability are dealt with by Siggia.[8]

Equation (2b) is the appropriate core model for collapse since, as we will see, the time necessary for $ds/d\theta \to \infty$ is comparable to the characteristic time for core rearrangements, σ^2/Γ. Furthermore, σ does not vary rapidly with arc length around the point of its minimum so that some redistribution of core volume along the filament would not matter. The core model in Ref. 8, $\sigma^2 L = \mathrm{const}$ (L = total arc length) is clearly inappropriate once significant stretching begins, yet three-dimensional pictures of the filaments look very similar to (2b).

To follow the collapse as far as possible with (2b), our code focuses down on the region with smallest σ. Thus we are unable to say if the total arc length, L, is infinite when σ first hits zero for some θ. However, because (2a) becomes nearly local in s, many values of θ should become singular in a finite time if one does. The singular set must clearly be thought of in space-time.

The most striking and surprising consequence of (2) is the pairing between oppositely directed sections of the vortex filament.[8] The spacing is of order σ, permanent, and independent of initial conditions. The filament pair must be thought of as a single dynamical entity. The pairing justifies the local model below (i.e., the nonlocal velocity falls off as a dipole) while the local terms conspire to maintain the pairing.

The stretching process is triggered by a version of the Crow instability appropriate to tightly paired antiparallel filaments.[9] Locally the paired filament looks like a small piece of a vortex ring together with its image in a parallel plane. The ensemble convects itself radially outward with a speed of order Γ/σ (recall that the spacing is $\sim\sigma$) and a differential stretching $\sim\Gamma/\sigma r_c$ where r_c is the local radius of curvature. After a certain amount of new line is produced the Crow instability is reactivated and smaller-scale folding begins. How all this leads to singularity in finite time will be apparent once we become more quantitative. (Nearly parallel filaments will remain so for some time and wrap around each other. The vortex stretching then occurs through differential rotation which is essentially a linear process and does not seem to yield finite-time singularities.)

We conclude the numerical discussion with a few details. Most significantly, the distribution in r_c scales with σ yet maintains $r_c/\sigma \gtrsim 4-5$ point by point except for $\sim 25\%$ of the instances and then only 5% of the points when the ratio fell to 2–3. The interfilament spacing decreases from 1.3σ to 0.6σ while σ^2 decreases by a factor of 200. The core size is uniform in arc length about its minimum to the extent that it does not exceed $1.5\sigma_{\min}$ over a $\Delta L \sim 100\sigma_{\min}$. Lastly, σ^2 tends to zero as t^*-t with corrections that could plausibly be fitted with logarithms. Figure 1 looks scalloped and slightly steeper than linear since we chose to plot $\sigma_{\min}$ irrespective of location to prove that some point hits zero in a finite time.

Further analytic reduction of (2a) is possible if σ/r_c is treated as a small parameter. A particularly simple model is obtained in terms of an average, R, and difference variable ρ for the filament pair (i.e., $r_{1,2}=R\pm\rho/2$),

$$\frac{d\mathbf{R}}{dt}=\frac{2}{\rho^2+2\sigma^2}\rho\times\frac{\partial\mathbf{R}}{\partial s},$$

$$\frac{d\boldsymbol{\rho}}{\partial t}=\frac{\partial\mathbf{R}}{\partial s}\times\frac{\partial^2\mathbf{R}}{\partial s^2}\ln[1+\rho^2/(2\sigma^2)] -\frac{2}{\rho^2+2\sigma^2}\rho\times\frac{\partial\boldsymbol{\rho}}{\partial s}\ , \tag{3}$$

where σ obeys (2b), $ds=|d\mathbf{R}|$, and $\Gamma/4\pi=1$. Furthermore if $\rho\,\partial R/\partial\theta=0$ initially, it remains so.

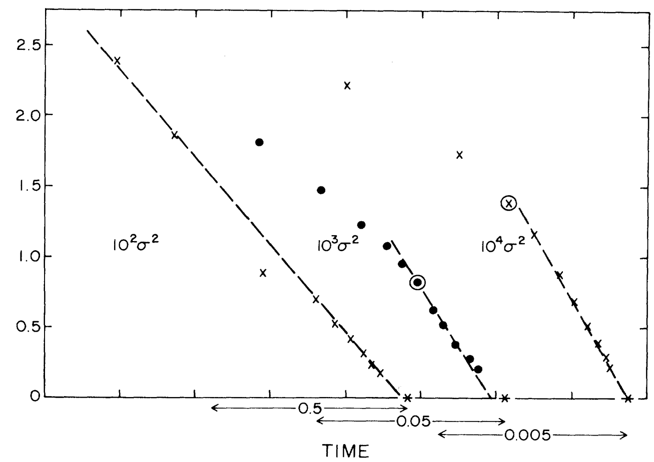

FIG. 1. The minimum core size squared vs time. Both scales are linear and magnified ten times for each of the three successive bands of data points (alternating crosses and circles). The asterisks mark a common value of t^* that has been shifted to separate the various scales. The horizontal arrows illustrate the magnification. The uppermost point in each of bands 2 and 3 repeat the last point in the previous band. The circled points show the times when the location of the minimum has jumped to another point along the filament.

With no further approximations we find for $y=\rho^2/(2\sigma^2)$

$$dy/d\ln(\sigma^{-2})=y-(1+y)\ln(1+y)\ , \tag{4a}$$

or

$$\rho^2\sim 2\sigma^2/[\text{const}+\ln(\sigma^{-1})]\ , \tag{4b}$$

when $\sigma\to 0$. The right-hand side of (4a) acts as a Liapunov function which "confines" the ratio ρ/σ when the "time," $\ln(\sigma^{-2})$, tends to infinity. Equation (4b) is consistent with the slow decrease in ρ/σ as $t\to t^*$ that we found numerically.

Two further assumptions, both supported numerically, yield some analytic understanding of why (2) diverges in a finite time, i.e., impose $\max|R|<\text{const}$ and $0>\text{const}\geq\hat{\boldsymbol{\rho}}\cdot\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the binormal to $\mathbf{R}$. Then ignoring logarithms,

$$L^{1/2}(t)>\text{const}\int_0^t L\ . \tag{5}$$

Equation (5) is a statement of convexity which forces a finite-time singularity. A further assumption that $r_c/\sigma\sim\text{const}$ implies, $\sigma^2=\text{const}(t^*-t)$, neglecting logarithms.

Equation (3) must be understood only as a step in the derivation of (4) and (5) rather than as a new model whose trajectories will track (2a). Certain small terms have to be included to make its linearized spectrum more closely match (2a) before it can reasonably be integrated forward in time. Equation (5) then appears merely formal since pointwise singularities with $r_c/\sigma\sim\text{const}$ and $\sigma^2\sim\text{const}(t^*-t)$ occur with L finite.

The small value of σ/r_c that we found for solutions to (2a) and (2b) is the key for building from them solutions of the Navier-Stokes or Euler equations by multiscale perturbation techniques. To restore the correct core structure, one begins from finite-core vortex solutions to the two-dimensional (2D) Euler equations subject to axial stretching, and scales them with $\sigma(\theta)$ as one moves along the filament. The expansion around $\sigma/r_c=0$ is systematic.

The extreme stability of vortex dipole solutions in the 2D Euler equations has been demonstrated many times numerically.[10] The third dimension enters perturbatively to lowest order in σ/r_c as a uniform axisymmetric strain and as such can be completely accounted for by a scale change even for the Navier-Stokes equations. Let $\lambda(t)=(ds/d\theta)^{1/2}$ be the integrated strain; then if lower-case variables denote the strained system the rescaled variables are,

$$(\boldsymbol{X},\boldsymbol{Y})=(\lambda x,\lambda y)\ , \tag{6a}$$

$$T=\int_0^t \lambda^2 dt\ , \tag{6b}$$

$$\Omega(\boldsymbol{X},\boldsymbol{Y},T)=\lambda^{-2}\omega(x,y,t)\ . \tag{6c}$$

The viscosity is unaffected. An approximate 3D solution, ω, is thus obtained from (6c) by imagining x,y to be perpendicular to the filament and adjoining $\lambda(t)$ computed from the Biot-Savart equations.

We consider how higher-order terms could modify the core shapes and whether the inclusion of viscosity will cause the evolution of the 0th order solution to deviate from the Biot-Savart formula. The former question is a delicate one since the 3D energy computed from quasi 2D solutions scales as $\Gamma^2 L$ (if we assume constant shape). Hence when one region amplifies, energy must be drawn from the large scales elsewhere in the fluid. The collapse that we observe is a very efficient means of transferring and focusing energy.

To bound the distortion crudely we examined the order of various corrections in σ/r_c and assumed that any velocity thereby generated acted monotonically in time to change the shape. Even so, perturbations only became appreciable for $\ln[\sigma(0)/\sigma(t)] > \sim (r_c/\sigma)^{1/2}$. The actual dynamics of the collapse entered our estimates. When a vortex ring approaches a free slip wall and expands radially, the arc length grows only as t^2 and the core shapes can change by $O(1)$ when the radius doubles. In fact the cores must deform if the ring remains axisymmetric so as not to violate energy conservation.

Clearly σ/r_c, though formally small, may not be small enough in actuality. Several runs were restarted with one filament replaced by a bundle of four and no secular degradation of the cores was observed. Since we are uncertain whether $L/L(t=0)$ is large when σ^2 first hits zero, it is worthwhile to note that (3) suppresses certain instabilities of (2) and thereby makes evident that (2) admits unstable solutions with $\sigma\to 0$ and little growth in L.

While (6) also provides the essential connection between solutions of the Biot-Savart and the Navier-Stokes equations, it is informative to give a more heuristic argument as to why the strain can nearly overwhelm the viscous diffusion. Consider the most naive modification of (2b) that includes the viscosity,

$$\frac{d\sigma^2}{dt}=\nu-\sigma^2\frac{d\ln(ds/d\theta)}{dt}\ . \tag{7}$$

Either from (3) or directly from the flow field for a pair of curved vortex tubes, the strain rate in (7) scales as the velocity, $\Gamma/\max(\rho,\sigma)$ divided by the radius of curvature. Since both r_c and ρ scale with σ, the right-hand side of (7) is $\nu-\text{const}\Gamma$ which may be negative allowing σ^2 to vanish. For these reasons we are unable to imagine any other flow in which the self-stretching balances the viscosity.

It is now also apparent why if we modified the viscous term in the Navier-Stokes equations by $\tilde{\nu}\nabla^{2+\epsilon}$ and replaced ν by $\tilde{\nu}/\sigma^{\epsilon}$ in (7), no singularity is possible. In a real sense the Navier-Stokes equations almost diverge because the viscosity and circulation have the same units.

Our discussion up to this point has assumed that Γ is fixed and unaffected by viscous diffusion. Recall (6b) and imagine that $ds/d\theta \sim 1/(t^*-t)^{\alpha}$, then for $\alpha<1$, T is finite at t^* and at most a fraction of Γ will be lost which changes nothing. Conversely for $\alpha>1$, $T(t^*)$ is infinite and diffusion may entirely eliminate the vortex dipole. The actual value either numerically or from (3) is $\alpha=1$ plus logarithms. This is the minimum α that would satisfy Leray's bounds[1] and is another manifestation of the marginal nature of the usual Laplacian formula for the damping.

Some degree of stability is necessary in order for our solutions to have any experimental relevance. However, isolated points where the pair pinches off should not alter our conclusions. It may be that core instabilities disrupt the initial pairing process studied in Ref. 8, yet would not destroy the collapse after pairing occurred. For this reason, wall-bounded shear flows are of interest since paired filaments ("hairpins") are known to occur.[11]

Our numerical simulations plus analysis of (3) convince us that the Biot-Savart equations lead into infinite stretching in a finite time. The biggest uncertainty in promoting even nearly straight vortex filaments into solutions of the Navier-Stokes or Euler equations are purely inviscid effects that would turn the cores into ribbons as they stretched. Ribbons would be unstable in spite of any stretching. If the cores remain circular, then we have a good argument why viscosity is only marginally able to control the divergence. The linear decrease of σ^2 figures essentially in the demonstration.

Our results fall short of common expectations for the Euler equations, even granting that vortex tubes follow the Biot-Savart equations. We claim, for instance, only that $\max|\omega|$, not the enstrophy, blows up in finite time.

There may be no universality with respect to initial conditions, e.g., max $|v|$ may never diverge for the Taylor-Green flow; nevertheless, we resist any suggestion that vortex tubes are more singular than Fourier modes. Whatever size the core has initially merely fixes the time scale and has no other dynamical significance.

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[1]J. Leray, Acta. Math. **63**, 193 (1934).

[2]V. Scheffer, Commun. Math. Phys. **61**, 41 (1978).

[3]L. Caffarelli, R. Kohn, and L. Nirenberg, Commun. Pure Appl. Math. **35**, 771 (1982).

[4]D. W. Moore and P. G. Saffman, Philos. Trans Roy. Soc. London, Ser. A **272**, 403 (1972).

[5]J. L. Lions, *Quelques Methods de Resolution des Problems aux Limits Nonlineaires* (Dunod-Gauthier-Villard, Paris, 1969); B. Nicholis, private communication.

[6]M. E. Brachet, D. Meiron, S. Orszag, B. Nickel, R. Morf, and U. Frisch, J. Fluid Mech. **130**, 411 (1983).

[7]A. J. Chorin, Commun. Math. Phys. **83**, 517 (1982).

[8]E. D. Siggia, Phys. Fluids **28**, 794 (1985).

[9]S. C. Crow, AIAA J **8**, 2172 (1970).

[10]H. Aref and E. D. Siggia, J. Fluid Mech. **109**, 435 (1981); J. C. McWilliams and N. J. Zabusky, Geophys. Astrophys. Fluid Dyn. **19**, 207 (1982).

[11]J. Kim, Phys. Fluids **28**, 52 (1985).