RESEARCH NOTES

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Point-vortex simulation of the inverse energy cascade in two-dimensional turbulence

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The conventional cloud-in-cell vortex representation of the two-dimensional Euler equations is modified to allow energy injection from small scales. A simulation of the inverse cascade yields an energy spectrum consistent with $k^{-5/3}$ and a Kolmogoroff constant of 14.

The two conservation laws of energy and enstrophy are well known to greatly constrain the manner in which an inviscid, statistically homogeneous and isotropic, two-dimensional flow can relax to equilibrium.[1] When a two-dimensional fluid is randomly forced, energy cascades to small wavenumbers and enstrophy to large wavenumbers.[2]

Far less attention has been paid to the inverse energy cascade than to the enstrophy cascade since Lilly's first numerical simulations.[3] In this note, we report on a simulation of only the inverse cascade, but on a considerably larger scale than Lilly was able to run. To accomplish this, we have extended the scope of point vortex methods in two dimensions by devising a physically plausible forcing technique to inject energy.

Vortex methods have been successfully used to simulate a number of nearly inviscid flows when the vorticity is nonuniformly distributed. When used in conjunction with a lattice to facilitate inversion of Poisson's equation (the cloud-in-cell algorithm[4]), their numerical efficiency is competitive with finite difference or spectral methods. For the statistically homogeneous, isotropic flow simulated here, one might hope that, in spite of the effects of the lattice, a vortex simulation would better express the local conservation of circulation (Kelvin's theorem). A second reason for applying vortex methods to this problem concerns the forcing. When a spectral or finite difference simulation is forced at some wavenumber k_f, the ratio of enstrophy to energy injected is given by k_f^2 (Ref. 3). The former must be dissipated by viscosity so the simulation must extend at least out to $4k_f$. If an enstrophy sink is not provided, the energy spectrum will remain peaked about k_f and no inverse cascade will materialize. As we shall see, vortex methods do not require a viscosity and the $-\frac{5}{3}$ spectrum may extend out to the highest wavenumber resolved.

Let ψ be a stream function obtained from the Poisson equation

$$\nabla^2\psi = -\omega\,, \tag{1}$$

where the vorticity $\omega = \sum_i \kappa_i \delta(\mathbf{r}-\mathbf{r}_i)$, $\sum_i \kappa_i = 0$, and each $\kappa_i = \pm\kappa$. Vortices move according to

$$\dot{\mathbf{r}}_i = -\hat{z}\times\nabla\psi'(\mathbf{r}_i) + \mathbf{v}_f(\mathbf{r}_i)\,, \tag{2}$$

where ψ' is the stream function due to all vortices except the one at $\mathbf{r}_i$ and the last term represents the as yet unspecified forcing. The total kinetic energy, less the self-energy of the vortices, is $1/2\sum_i \kappa_i\psi'(\mathbf{r}_i)$. We invert (1), subject to periodic boundary conditions, after smoothing the vortices onto a lattice. The spatial derivatives in (2) are computed as centered differences on the lattice and then interpolated. (While our code generally follows Ref. 4, additional details and diagnostics are given in Ref. 5).

In the atmosphere, the energy supplied to the largest scales ultimately comes from small scale processes such as convection. If, however, the motions of interest are separated from the actual forcing by a larger range of scales than can be accommodated on the computer, some subgrid modeling is needed. The "force" then represents energy transfer from smaller scales. Within the context of a spectral simulation, closures could

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provide the necessary parametrization for the energy transfer from small scales,[6] but the rather intricate coding involved has never been done. Instead, we shall suggest a physically plausible functional form of the forcing to be used in conjunction with (2). (Its appropriateness could be checked by comparing simulations covering different ranges of scales just as is done for subgrid models in three dimensions.[7])

Let us assume

$$\mathbf{v}_f(\mathbf{r}_i) = \beta\kappa_i \nabla\psi'_F(\mathbf{r}_i) , \tag{3}$$

where β is a scale factor (possibly time dependent) and ψ'_F is the stream function ψ' filtered to remove all wavenumbers less than some k_f. While the strongest arguments in favor of (3) can be given *a posteriori*, it can be made physically plausible.

In the absence of forcing, Eq. (2) represents the advection of vortices along the instantaneous stream lines. The force (3) is provided by a small incremental velocity up or down the local stream function gradient depending on the sign of κ_i. Since the energy is itself the product of a vortex strength κ_i, and a stream function ψ', it is not implausible that (3) will increase the energy. Equation (3) is also in accord with the notion of negative viscosity. Individual vortices are pushed toward clusters of like sign contrary to the action of ordinary diffusion. Lastly, imagine that for some reason (e.g., drag against a substrate) the motion of a point vortex lags behind the local velocity. The Magnus force will then result in a component of motion at right angles to the local velocity. Equation (3) points in the opposite direction of this incremental velocity.

The small k components of ψ' were filtered out to arrive at (3) since we insist that the energy input is unchanged if a large parcel of fluid is advected uniformly. That is, the energy transfer from small to large scales is not modified by uniform sweeping.

The data we present were obtained with $128^2 = 16384$ vortices on a 256×256 lattice. Extensive runs were also made on coarser lattices with proportionally fewer vortices where the dependence on various parameters could be explored and various possible systematic errors examined.

The energy spectra shown in Fig. 1 were computed from the Fourier components of the lattice vorticity. All modes with integral wavenumber $k < 64$ were removed from ψ'_F in (3), where units are used in which the minimal allowed wavenumber is 1 and $\beta|\kappa| = 0.004$. The forcing is localized as much as seemed sensible to high k. Along with Fig. 1 we prepared a plot of energy vs time which was linear to within a few percent and the slope of which defined ϵ, the rate of energy input.

Initial conditions were chosen to approximate statistical equilibrium[1] with as much energy as possible at high k. Once the high k modes adjust they remain fixed and energy cascades backward. The rate of increase of enstrophy (computed from the lattice vorticity) decreases since ϵ is constant and energy is being added at progressively smaller wavenumbers. The Kolmogoroff constant is of order 14. When a larger value of β was used, the entire spectrum moved up before the inverse cascade developed. With much smaller β a hump in $E(k)$ at large k remained from which the $-\frac{5}{3}$ spectrum emerged. In either case the Kolmogoroff constant was indeed constant with changes in $E(k)$ and ϵ compensating.

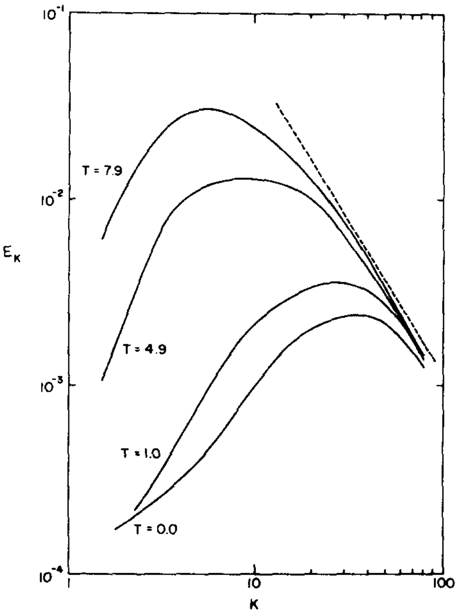

FIG. 1. A log–log plot of the energy spectrum $E(k) = \pi k\langle \mathbf{v}_k \cdot \mathbf{v}_k\rangle$ at four successive times. The dash-line has a slope of $-\frac{5}{3}$.

For the later times we were able to demonstrate that the rate of energy increase of the large scales was correctly predicted by computing the energy transfer from the two-dimensional Euler equation using ψ defined on the lattice. Thus, whatever defects Eq. (3) may have, it has been shown to lead to a constant rate of energy input localized at high k. Having done this, we have not addressed the more delicate question of how universal are the statistics of the largest scales. A more subtle dependence on the forcing function rather than just through ϵ is conceivable.

The most important conclusion to be drawn from our simulation is the unexpectedly large Kolmogoroff constant, which is about twice the accepted value.[3,8] A large Kolmogoroff constant is indicative of an inefficient cascade; that is a small ϵ is associated with a given $E(k)$ or local shear, or a large shear is required to generate a given ϵ. It is appealing to conjecture that a vortex code gives a more accurate account of the local dynamics in two dimensions and thereby impedes the energy transfer. It would obviously be interesting to repeat our simulations with a spectral code of comparable spectral range.

The advection and forcing effects are on a very equal footing in Eq. (3). Once $E(k)$ has stabilized for large k, the ratio $|\mathbf{v}_f|/|\nabla\psi'|$ never exceeds 0.003 and is decreasing. The motion of the vortices thus is very nearly coincident with the instantaneous stream lines. A Kolmogoroff constant of order 10 is really very large. One might therefore speculate that intermittency will not be significant in the inverse energy cascade and the modal statistics will be nearly Gaussian.

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Wavenumber variations of spatially damped Rayleigh–Bénard convection

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Under subcritical conditions, it is shown that a spatial variation of the velocity amplitude of convective rolls is accompanied by a modification in their wavenumber which takes values lower than the critical one.

The question of the analogy between second-order phase transitions and hydrodynamic instabilities as exhibited, for instance, by the Rayleigh–Bénard problem has received much attention in recent years. In particular, Wesfreid *et al.*[1] found that the convection induced at one side of a Rayleigh–Bénard box maintained under subcritical conditions exhibits a spatial exponential decay of the amplitude of the rolls with a characteristic length ξ_- varying as $\epsilon^{-1/2}$, where $\epsilon = (R_c - R)/R_c$, R and R_c being the imposed and critical Rayleigh numbers. The numerical simulation of such a triggered subcritical convection in the electrohydrodynamic case[2] further revealed a decrease in the wavenumber k when R decreases below R_c, a feature also visible from a similar study in the Rayleigh–Bénard case.[3]

Here, we focus on this wavenumber variation with the idea that a spatial modulation of the amplitude of the rolls is necessarily accompanied by a variation in the wavenumber. In this view the problem might have some relevance to the supercritical conditions ($R > R_c$) where a decrease in k vs R has been observed.[4]

Let us return to the Rayleigh–Bénard problem under subcritical conditions ($R < R_c$) with a triggered convection roll of low enough amplitude so that the z component of the velocity w satisfies the linear equation

$$\nabla^6 w = R \nabla_1^2 w . \quad (1)$$

Seeking for an asymptotic solution of the form

$$w(x,z) = W(z)\exp(ikx - \alpha x) ,$$

Eq. (1) gives

$$(D^2 + \gamma^2)^3 W = R\gamma^2 W , \quad (2)$$

where $D = d/dz$ and $\gamma = ik - \alpha$. In the case of two horizontal free surfaces, the solution of (2) with the corresponding boundary conditions is $W(z) = \sin \pi z$ which leads to

$$(\gamma^2 - \pi^2)^3 = R\gamma^2 . \quad (3)$$

Since R is real, Eq. (3) gives two equations (relative to real and imaginary parts) relating the two variables α and k. For $R < R_c$, these equations have a unique couple of real roots ($\alpha > 0$). Figure 1(a) shows that k increases with R ($R < R_c$). For $R \to 0$, we can deduce from (3) the asymptotic expressions

$$k \simeq (\sqrt{3}/4)(R/\pi)^{1/3}, \quad \alpha \simeq \pi - \tfrac{1}{4}(R/\pi)^{1/3} .$$

For $R \to R_c (\epsilon \to 0)$, it is easy to see that k and α^2 vary linearly with ϵ. Putting

$$k = k_c(1 + a\epsilon), \quad \alpha^2 = k_c^2 b\epsilon ,$$

with $k_c = \pi/\sqrt{2}$, we obtain, after substitutions, $a = -7/24 = -0.292$ and $b = \frac{3}{4}$. The formula obtained, $\alpha = \sqrt{3/2} \times (\pi/2)\epsilon^{1/2}$, is identical to the law for the influence length $\xi_- (\xi_- = \alpha^{-1})$ given by Wesfreid *et al.*[5] for the free-

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