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SIMULATIONS OF INCIPIENT SINGULARITIES IN THE 3-D EULER EQUATIONS

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The existence of singular solutions in the 3-D Euler equations is considered. A numeric algorithm for simulating collapsing solutions is described, and preliminary results are presented.

The serious study of singularities in the equations of three-dimensional hydrodynamics goes back at least to 1934, and a proof still has not been given that solutions of the Navier–Stokes equations remain smooth [1]. Far less is known rigorously about solutions to the Euler equations. Although on physical grounds one might expect a singularity here, the numerical evidence is unpersuasive.

Other than being a challenging mathematical problem, singularities may furnish a paradigm for three-dimensional flows. An analogy should be drawn with two-dimensional shear flows, specifically the mixing layer, where the first visualization studies revealed large vortex blobs [2]. Their persistence came as a complete surprise since this same flow had been studied earlier with hot wires and generally found to evolve in accordance with simple mixing length or dimensional scaling ideas. The coexistence of simple scaling and coherent structures has subsequently been verified in other 2-D flows [3].

The analogy we wish to draw for 3-D is between coherent structures in space and those in space–time; or singularities. If we take a turbulent boundary layer as the prototypical 3-D flow, we are forced to consider the bursting phenomena which accounts for an appreciable fraction of the Reynolds stress [4]. Boundary layer bursts also came as a surprise since the mean velocity profile was known to obey the von Karman scaling law. There is still debate about the proper interpretation of bursts but there is incontrovertible evidence of very large gradients on very small scales [5].

Two further more technical arguments for studying singularities should be advanced. Some form of collapse is the only way in which a computer can simulate a large range of scales. Under favorable circumstances, the simulation time will grow logarithmically with scale size rather than algebraically. Analytic calculations may also prove feasible for a collapse and point the way towards more generally applicable approximations.

Our numerical approach to singularities should be contrasted with two previous efforts, both inconclusive. Brachet et al. [6] computed a power series in time for the enstrophy with Fourier modes, which was then Padé approximated. A subsequent theorem showed this analysis to be superfluous. Namely, Beale et al. proved that the enstrophy can only diverge if the time integral of

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the supremum (over space) of the vorticity does [7]. In ref. [6] this quantity grew by no more than a factor of 4–5, hence no singularity.

One can also question whether Fourier modes are the optimal way of describing spacial singularities. The equations are local in real space except for the pressure which would tend to average out for a singularity with fractal support. Our working assumption is that the first singularity is point-like, though more complicated objects could develop later.

Chorin applied a vortex algorithm to this problem but in our view its implementation was inadequate [8].

Pumir and Siggia solved the Biot–Savart model for a single vortex filament with a locally variable core size chosen to preserve volume [9]. A singularity was found, which modulo logarithmic terms, would persist with viscosity present. This model is perhaps the simplest possible which contains vortex stretching and can be solved with reasonable confidence both numerically and analytically. It invalidates any casual arguments that viscosity will control a singularity in the Euler equations.

This model has one serious shortcoming which was explicitly noted in ref. [9], namely that the vortex cores undergo secular distortion when they pair and stretch [10]. Vortex reconnection is not an issue since it clearly does not occur for Euler prior to the singularity, and for Navier–Stokes, large gradients diverging with the inverse viscosity are necessary for reconnection to occur in a finite time. Granting that the collapse does preserve some core shape, then it is of no importance whether the shape is circular or not. It is necessary to adjust the core size locally since comparison with the correct equations shows that there is insufficient time for the core volume to redistribute. Lack of manifest energy conservation is also not a problem since under these assumptions the energy in the Euler flow to which the filaments are asymptotic is finite at the singularity but has a square root cusp in time. The distortion however can be viewed as the consequence of an insufficient energy flux into the singular region.

To simulate a 3-D singularity without modeling in the absence of boundaries, we have mapped the line onto the interval with the tangent function and then finite-differenced the Euler equations in each direction separately. The code is second order accurate in space and time and explicitly respects incompressibility, and momentum and energy conservation. We initially adjusted scales continuously to preserve certain norms, but found this both to interfere with the conservation laws and to bias the result through the choice of norms. Currently, we periodically interpolate from the old mesh to a new one using splines under tension.

If we initialize this code with two antiparallel vortex tubes, designed to imitate the Biot–Savart model, the tubes first press together and flatten into sheets or ribbons with the direction of vorticity and its magnitude changing little. Vortex amplification then ensues at the leading edge of the pair of ribbons which assume a V shape. The eigenvalues of the rate of strain tensor have the signs $(+,+,-)$ with the compression pushing the ribbons together and the largest expansion along the direction of propagation. The smaller expansion is along the vorticity direction.

It is not yet clear whether the maximum vorticity diverges in a finite time. The enstrophy may well not have a finite time singularity. The velocity grows much more slowly in time than the maximum vorticity in contrast to the Biot–Savart results. The product of the spacing between the ribbons and the maximum vorticity is roughly constant. The flow also appears to be self-similar.

The best qualitative account we can give of the fluid mechanical origins of the vortex self-stretching in this flow is by reference to the Biot–Savart simulations. In spite of qualitative differences, the relation of vorticity and strain is the same, as well as the resultant velocity.

The numerical calculation is basically not a difficult one since there is one dominant scale in each direction and the flow is smooth. The power of our method stems from the successive mesh refinements which are common enough in other contexts but have not yet been employed on the

3-D Euler equations. If a singularity exists for our initial conditions, we expect that our simulations will exhibit it and provide strong guidance for an analytic perturbative treatment of the collapsing solution.

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