PAINLEVÉ PROPERTY AND GEOMETRY

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The Painlevé property for discrete Hamiltonian systems implies the existence of a symplectic manifold which augments the original phase space and on which the flows exist and are analytic for all times. The augmented manifold is constructed by expanding the Hamilton–Jacobi equation. A complete classification of the types of poles allowed in complex time is given for Hamiltonians which separate into the direct product of hyperelliptic curves. For such systems, bounds on the degrees of the (polynomial) separating variable change, and the other integrals in involution can be found from the pole series and the Hamilton–Jacobi equation. It is shown how branching can arise naturally in a Painlevé system.

1. Introduction

The study of exactly solvable or integrable dynamical systems has provided valuable insights into problems lacking closed form solutions as well as unexpected connections between geometry and analysis. While it is possible to prove a system is not integrable, say by exhibiting a homoclinic point, there seems to be no general way to prove integrability except by explicitly producing integrals. Numerical methods can be of assistance, but the need for an analytic test has long been appreciated.

Kovalevskaya [1] was the first to search for integrable examples by assuming they would have singularities no worse than poles when continued to complex times. Painlevé systematized this type of analysis and found all equations within a certain class with this property, (plus fixed essential singularities). The connection between what is now called the Painlevé property and integrability was only made quite recently [2]. The essential correctness of the Painlevé–Kovalevskaya procedure has been demonstrated in many examples [3–7]. Weiss and others have used an extension of this analysis to calculate key properties of many of the canonical soliton equations [8, 9].

Integrability places strong restrictions on the geometry of flows in phase space. The above referenced papers are largely analytical and local in content, while geometric constructs are global and more qualitative. Our aim in this paper is to show how more intensive use of the geometry of phase space enhances the power of the Painlevé test and adds to its plausibility. Another geometric approach to this problem has been undertaken by Adler, van Moerbeke, and Haine [6, 7, 10].

In this paper we will restrict our attention to systems of differential equations on a complex Euclidean space C^{2n}, of even dimension, which are Hamiltonian. In conjugate variables $\{q_i, p_i\}_{i=1}^{n}$ on C^{2n} these have

the form

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i},$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i},$$

(1.1)

where $\mathcal{H} = \mathcal{H}(p,q)$ and is independent of time. We will further assume that $\mathcal{H}$ is a polynomial in these variables.

The results of this paper fall into two categories:

(i) Construction of a partial compactification of the phase space, C^{2n}, for eqs. (1.1) which have the Painlevé property (defined below). We call this an *augmentation* of C^{2n}; it provides "coordinates at infinity" which regularize the blow-up of solutions and permit their continuation for all time.

(ii) Several approaches to bounding the degrees of *polynomial integrals* for a system (1.1), if the system is in fact integrable. Precisely, we provide an algorithm for deciding whether or not a system is *separable* in the sense of Jacobi.

We will give a fuller description of our results, and the layout of this paper, after developing some background.

1.1. Basic concepts

Our results are framed in terms of the singularity analysis of solutions to (1.1). We briefly recall the ingredients which we will use.

One constructs formal Laurent series solutions

$$q_i = t^{-f_i}\left(\alpha_0^i + \alpha_1^i t + \cdots\right), \quad \alpha_0^i \neq 0,$$

$$p_i = t^{-g_i}\left(\beta_0^i + \beta_1^i t + \cdots\right), \quad \beta_0^i \neq 0$$

(1.2)

by direct substitution of these series into (1.1). These solutions may depend on up to $2n - 1$ free parameters. (We have suppressed a parameter, t_0, corresponding to time translation which can be restored by substituting $(t - t_0)$ for t in (1.2).) Each parameter can be assigned an order, ρ, which is defined by the coefficient, $\binom{\alpha_\rho}{\beta_\rho}$, where the parameter first appears. For a given formal solution, (1.2), these numbers ρ, counted with multiplicity, are referred to as the *resonance orders or degrees* of the solution.

We can now define some terminology.

Definition 1. A formal solution (1.2) of (1.1) with at least one diverging (q_i) or (p_i) is called a *balance*. The numbers (f_i, g_i) are called the *leading exponents*.

A balance is *principal* if it depends on $(2n - 1)$ free parameters $(t_0$ excluded). Balances with fewer parameters are *lower* and may be ordered by the number of parameters present.

Definition 2. The system (1.1) is Painlevé if the solution is meromorphic in the vicinity of any singularity.

Remark 1. We need not restrict ourselves to "movable" singularities, (those whose position depends on initial data), in definition 2 as is conventionally done, since our equations of motion are polynomial, and therefore have no fixed singularities.

Remark 2. For any Painlevé system, the leading exponents and resonance orders must all be integers. Also singularities can only occur when some variable blows up, again because (1.1) is polynomial. To show a system is Painlevé, once the formal balance series are known, requires a convergence proof which follows from the result of section 2.

Remark 3. We consider as Painlevé systems whose solutions are entire in time. Actually, the considerations in section 2 suggest that if one initial point (q, p) blows up in a finite time, so will an open set of points containing it. In all examples we are aware of, only codimension one or higher sets of initial data remain entire; however, a general proof is lacking.

We will have little to say here about the practical aspects of calculating the resonance orders ρ. Once consistent leading exponents and coefficients (α_0, β_0) are known, the allowed ρ may be found by linearizing (1.1) about the leading terms. An eigenvalue problem for the "Kovalevskaya" matrix results, whose non-negative solutions are the desired ρ. Actually, the Hamiltonian form of (1.1) imposes additional structure on the set of resonance orders if they are defined more precisely. The necessary details and their consequences are left for the appendix as is an illustration of certain subtleties that extracting ρ from the linearization may entail.

1.2. *Augmentation*

The use of flows to construct and complete a space has appeared in many geometric contexts ranging from Morse theory to moduli spaces. A simple example, though not exactly of type (1.1), will serve to illustrate how our augmentation is built from flows.

The Riccati equation is

$$\dot{x} = a_2 x^2 + a_1 x + a_0,$$

where the a_i are entire functions of t. The dependent variable x is analytic whenever it exists and has only first order poles. Define $\tilde{x} = 1/x$ and observe that $\dot{\tilde{x}}$ also has Riccati form; $\tilde{x}(t)$ is analytic whenever it exists. The original coordinate domain $\{x \in C\}$, augments to

$$M = \{x \in C\} \cup \{\tilde{x} = 0\}.$$

The "point at infinity" $\tilde{x} = 0$ lies in an open coordinate domain $\{\tilde{x} \in C\}$. These two neighborhoods, together with the transition function $\tilde{x} = 1/x$ on the set where x, $\tilde{x}$ do not vanish, give M the structure of a manifold; in fact, M is just the Riemann sphere.

The systems we consider entail more than a one point compactification. The general augmentation involves adding a stratified set to C^{2n}; that is a disjoint union of manifolds (strata) possibly of varying dimensions. The different strata correspond to different classes of asymptotic solutions. In fact these strata and their relative configuration provide a complete geometric picture of the range of singularity behaviour of (1.1).

The algorithm which constructs the augmented phase space recasts in a very detailed way all the local analysis that led to the formal series (1.2). Section 2 formalizes these remarks. Our method associates a coordinate neighborhood (called a *patch*) to each stratum and a transition function between this patch and the original coordinates. This data is constructed by a *canonical transformation* which is generated by an asymptotic solution of the *Hamilton–Jacobi* equation

$$\mathcal{H}\left(\frac{\partial S}{\partial q}, q\right) = E \tag{1.3}$$

associated with (1.1).

The use of pole series to complete invariant sets of Painlevé systems has already received extensive treatment in particular cases. In particular, for two degree of freedom, Liouville completely integrable systems, Alder, van Moerbeke, and Haine [6, 7, 10] use pole series together with the polynomial integrals to embed a level set into a large ambient projective space where, by methods of algebraic geometry, they show that it completes naturally to an Abelian surface. Also Okamoto [13] and later Miwa and Jimbo [14] studied the two-dimensional systems classified by and named after Painlevé. By a sequence of σ-processes (better known as blowing up and blowing down) they construct an extended phase space for the flows.

Our approach, besides being different in method from previous treatments, has greater scope. We require no special properties for the system (1.1) other than that it be Painlevé and polynomial. No use is made of other integrals so we can handle Painlevé systems without regard to whether or not they are integrable. The construction is intrinsic, it is not necessary to work through an ambient space. Also the method is algorithmic and terminates in a prescriptible number of steps. Furthermore the coordinate patches at infinity produced by this procedure have a natural symplectic structure. Hence, it is easy to write Hamilton's equations in the new coordinates.

1.3. *The search for separable systems and their integrals*

In sections 3 and 4 we relate the concept of a Painlevé system to that of an integrable system in the sense of Liouville. Explicitly, we do this for systems in which the Hamilton–Jacobi equation, (1.3), has a solution, S, which separates under a change of variables, $q = q(\xi)$, into a sum of identical terms, i.e.,

$$S = \sum_{i=1}^{n} \int^{\xi_i} \left(P(\xi_i; h_1, h_2, \dots, h_n) \right)^{1/2} d\xi_i, \tag{1.4}$$

where the h_i are parameters and P is a polynomial in all its arguments. Under these assumptions it is possible to be very precise about the number and types of balances (see section 3).

As an example of our results, we show that when the variables (p_i, q_i) are given degree weights equal to the leading exponents (f_i, g_i) of a principal balance (these exponents are unique even if there is more than one principal balance), then

$$\epsilon \overline{\deg} \, \mathcal{H}_1(\boldsymbol{p}.\boldsymbol{q}) = 1 + \sup(f_i + g_i),$$

where $\overline{\deg}$ means "weighted degree" in the above sense and $\mathcal{H}_1$ is *the* integral of *lowest* weighted degree, and $\epsilon = 1$ or 2, depending on whether there are 2 or 1 principal balances. (Because of the hyperelliptic form of (1.4) there can be *at most* two principal balances.) Moreover, having found the first integral, $\mathcal{H}_1$,

the degrees of the other polynomial integrals are given by

$$\overline{\deg}\,\mathscr{H}_j \cdot \epsilon = \rho_j(\mathscr{H}_1),$$

where $\rho_j(\mathscr{H}_1)(\rho_i < \rho_j$ for $i < j)$ are the resonance degrees of a *lowest* balance of the $\mathscr{H}_1$-flow.

Given such bounds, the search for integrals, and hence separability, reduces to a finite search. In section 4 we also present an alternative formal scheme for reading off integral bounds from an asymptotic expansion of the Hamilton–Jacobi equation.

All these methods are in the nature of solving an inverse problem. Thus one requires a thorough characterization of the direct problem; that is of separable systems*. This we do in section 3. Some of this development overlaps existing results (in particular, see Adler–van Moerbeke [7] when $n = 2$). What is new here is:

(i) application of Painlevé analysis to the ξ-odes (i.e. the system (1.1) in separated variables) to give a systematic description of *all* balances;

(ii) use of Hartogs' theorem to show that for *separable* (1.1) with polynomial $\mathscr{H}$, there exists a complete set of integrals in involution with $\mathscr{H}$ which are polynomial.

(iii) a complete description of a new separable hierarchy, the Hénon–Heiles systems, which exhibits a locally separable variable change that is globally a *finite covering* of phase space.

As an extension of the results in section 3, we were able to explain a class of examples constructed by Ramani–Dorrizi–Grammaticos [5], which are separable but not Painlevé. Rather, they admit asymptotic solutions which are developed in fractional, non-integral, powers of $(t - t_0)$. In fact, these turn out to be projections, of the separable, Painlevé systems of section 3 onto lower dimensional sets. The level sets of these separable systems are not Liouville tori but rather a certain symmetric product of a Riemann surface with itself which has a complicated topology. The explanation of these examples is developed in the conclusion.

2. Phase geometry of a Painlevé system

2.1. *Augmented phase space*

We will now introduce the notion of "adding points at infinity" on the phase space C^{2n} of (1.1). It is actually natural to define this for an arbitrary autonomous system of ordinary differential equations (ode):

$$\dot{x} = F(x), \quad x \in C^m, \tag{2.1}$$

with F entire, analytic on C^m. Eq. (2.1) has the Painlevé property if the solutions

$$x = x(t - t_0; x_0) \tag{2.2}$$

have singularities which are at worst poles.

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*Actually, we place some technical conditions on our definition of separability. A precise definition is given at the start of section 3. Some of these conditions are made for simplicity and could be removed, while others are essential. However, the definition is broad enough to include all separable systems of type (1.1) that have so far been discovered.

It follows from standard ode existence theory that if U is a relatively compact subset of C^m, there exists an open disc $\Delta \subset C$ such that

$$x_{t-t_0}(x_0) = x(t - t_0; x_0): \quad \Delta \times U \to C^m. \tag{2.3}$$

In other words x_{t-t_0} is a family of holomorphic maps on U parametrized by $t - t_0 \in \Delta$. It is a geometric consequence of the Painlevé property that the phase space of a Painlevé system can be augmented so that the flows exist for all time; i.e. we can replace the righthand side, C^m, of (2.3) by the augmentation, M, so that $\Delta = C$. Thus x_{t-t_0} becomes a 1-parameter group of holomorphic maps on M.

We now give an implicit but precise characterization of M.

Definition 2.1. The augmented phase space, M, is, if it exists, the unique complex manifold such that:
a. C^m is an open dense complex submanifold of M;
b. $M - C^m$ is a finite union of irreducible analytic hypersurfaces of M;
c. there exists a complete analytic flow

$$x: C \times M \to M,$$

which extends the "chunks" $\Delta \times U \to C^m$ of (2.3);
d. two orbits $\bigcup_{t \in C} x_{t-t_0}(x_0), \bigcup_{t \in C} x_{t-t_0}(x_0')$ either coincide completely or are disjoint;
e. (minimality condition) if $\tilde{M}$ is any complex manifold satisfying the above properties, then $M \subseteq \tilde{M}$ is a complex submanifold. (This condition is included to make M unique.)

For a Painlevé system we propose to construct M as a union of coordinate patches which consist of $U_0 = C^m$, the original phase space and a patch $U_i \subseteq C^m$ ($i \geq 1$) for each balance (as defined in section 1). Let $\mathcal{U} = \{U_0, U_1, \ldots, U_\omega\}$ denote this open cover.

For $i \neq j$, consider $U_i \cap U_j$, an open subset of U_i. To construct a complex manifold M from $\mathcal{U}$ we must define biholomorphic maps or *transition functions*

$$\phi_{ij}: U_i \cap U_j (\subseteq U_i) \to U_j \cap U_i (\subseteq U_j) \tag{2.4a}$$

such that

$$\phi_{ij}^{-1} = \phi_{ji} \tag{2.4b}$$

and

$$\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = id \text{ on } U_i \cap U_j \cap U_k = (U_i \cap U_j) \cap (U_i \cap U_k). \tag{2.4c}$$

In effect the ϕ_{ij} "glue" U_i to U_j along the open set $U_i \cap U_j = U_j \cap U_i$, (fig. 1).

We now show that M is unique if it exists. Suppose that M_1 and M_2 are augmentations that satisfy definition 2.1. $M_i - C^m$ is the hypersurface at infinity which the augmentation adds to C^m. Both M_1 and M_2 contain C^m as a common open subset. Thus, in the coordinate patch for the kth balance, U_k, we have the following identification, through this common C^m (see fig. 2):

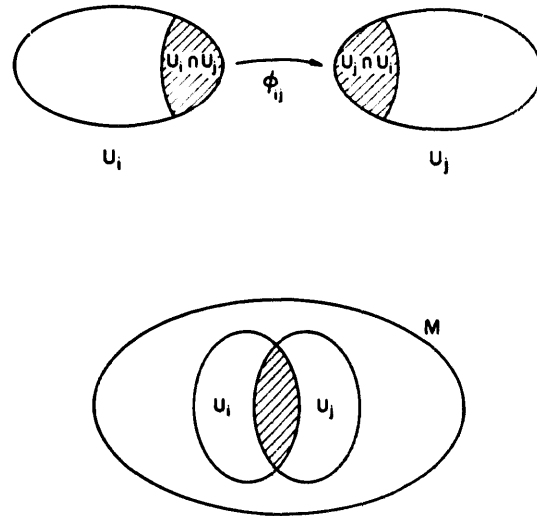$$U_k \backslash (M_1 - C^m) = U_k \backslash (M_2 - C^m), \tag{2.5}$$

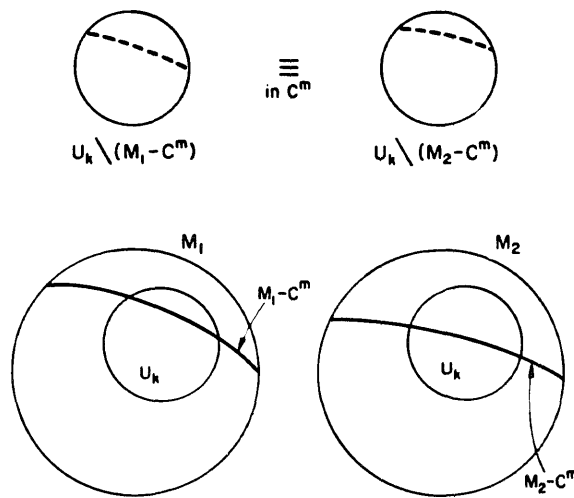Fig. 1. Patching of U_i, U_j by the transition function ϕ_{ij}.

Fig. 2. Two nominally distinct patches in M_1 and M_2 are identified through C^m. The dashed line in the upper pictures is $M_i - C^m$ and not part of C^m.

where $A \setminus B =_{df} A - (A \cap B)$. The idea of the argument is to use backward time flows, x_{-t_0} for various times t_0, to pull the respective hypersurfaces $M_i - C^m$ back into C^m where 2.5 can be used to identify the respective images. Fixing an arbitrary time t_0, condition (b) of definition 2.1 requires that $(M_i - C^m) \cap U_k$ is an analytic hypersurface in U_k. By (c) and (d) of that definition, the pull back $x_{-t_0}(M_i - C^m) \cap U_k$ is also an analytic hypersurface which is locally biholomorphic to $M_i - C^m$. Off of the codimension two set $(M_i - C^m) \cap x_{-t_0}(M_i - C^m) \cap U_k$, $x_{-t_0}(M_i - C^m) \cap U_k$ is the set of points in $U_k \setminus (M_i - C^m)$ which exit the original phase space in time t_0 (fig. 3). Hence $x_{-t_0}((M_1 - C^m) \cap U_k) = x_{-t_0}((M_2 - C^m) \cap U_k)$, since viewed from either M_i the two sets are subsets of C^m and may be identified trivially. Composing with the biholomorphic map x_{t_0} we get $(M_1 - C^m) \cap U_k \approx (M_2 - C^m) \cap U_k$ off of $(M_i - C^m) \cap x_{-t_0}(M_i - C^m) \cap U_k$. But then by varying t_0 we extend the isomorphism between $(M_1 - C^m) \cap U_k$ and $(M_2 - C^m) \cap U_k$ everywhere except subsets of $(M_i - C^m) \cap U_k$ which are invariant under the flow. However, such subsets
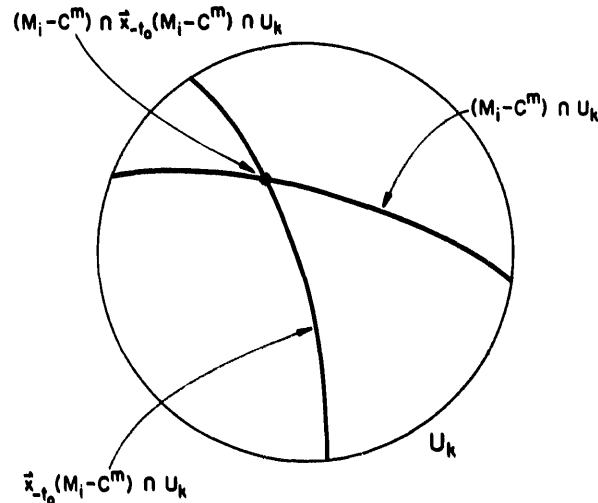
Fig. 3. Effect of the flow x_{t_0} on the hypersurface at infinity as seen in the patch U_k.

can be deleted from M_i without affecting conditions (a)–(d). Hence, since M_i satisfies condition (e), these sets cannot exist. Therefore $M_1 \approx M_2$.

We return, now, to the setting of Hamiltonian systems so that $m = 2n$. The above characterization provides a geometric interpretation for the balances. Since balances describe, at least formally, the ways in which solutions can blow-up, they correspond to what happens when the solutions "exit" the phase space C^{2n}. $M - C^{2n}$ consists of a locus of points added to C^{2n} which correspond in a one to one way with "places" where the orbits exit and return to C^{2n}. The topology of M is determined by how the p and q series blow up. Thus a principal balance is a $(2n - 1)$-dimensional sublocus of $M - C^{2n}$. In general, a lower balance depending on r parameters, excluding t_0, is an r-dimensional submanifold of $M - C^{2n}$. In section 2.2 we provide an effective procedure for constructing an augmentation, $\tilde{M}$, for a Painlevé system (subject to a mild technical constraint).

2.2. Painlevé completion of phase space

In this section we outline an algorithm to construct an augmented phase space M and in section 2.3, illustrate it with a number of examples. When lower balances are involved we are not able to demonstrate abstractly, using only the Painlevé assumptions (and a nondegeneracy condition), that all properties of M are satisfied. In particular it is very hard to show that certain consistency conditions on the transition functions between various patches are satisfied. Nevertheless in any example, the requisite checks are quite explicit. The Painlevé property and virtually all aspects of the pole series are exploited to build M. Even though we cannot prove, in general, that the existence of M follows from the Painlevé assumptions, the connection between the two is very tight.

With these reservations the construction of M proceeds in four stages.

(1) For each balance, one develops a corresponding formal expansion of the Hamilton–Jacobi equation for $S(q)$ which contains n free parameters if continued beyond a calculable order.

(2) A truncation $\tilde{S}$ of S then defines a canonical variable change from $\{q, p\} = U_0$ to a patch covering that portion of "infinity" corresponding to the balance in question. The patch variables are the n free parameters in $\tilde{S}$ and their conjugates.

(3) The Hamiltonian is rewritten in terms of the new patch variables and the flow extends unambiguously through infinity.

(4) The final and most complicated step is to derive transition functions among the patches that were added to cover infinity. If there are no lower balances, then all required transition functions were constructed in (2). Otherwise, starting from each principal balance, enumerate all lower balances in the closure of the principal balance submanifold in $M - U_0$. This is accomplished concretely by rewriting Hamilton's equations in the principal balance patch and seeking singular solutions which limit to the boundary region in question. All attached lower balances must occur this way and by repeating (1) and (2) transition functions follow.

Before discussing these four steps in more detail we work through a simple example consisting of the Weierstrass elliptic function.

The Hamiltonian is $2\mathcal{H} = p^2 - 4q^3 - aq$ where a is constant and p, q are canonically conjugate variables. There is precisely one principal balance,

$$q = \frac{1}{(t - t_0)^2}(1 + \mathcal{O}(t - t_0) + \cdots),$$

$$p = \dot{q},$$

and thus one patch to add. There is just one resonance with $\rho = 6$ corresponding to the energy.

The most expeditious way to integrate the Hamilton–Jacobi equation is to exploit the fact that there is only one degree of freedom to rewrite $E = \mathcal{H}(\partial S/\partial q, q)$ as

$$S = \pm \int (2E + aq + 4q^3)^{1/2} \, dq.$$

The integral can be expanded for large q up to at least order $q^{-1/2} \sim (t - t_0)$ so as to capture the dependence on E. More generally one has to inspect the Laurent series for p and q and use the equation $p = \partial S/\partial q$ to infer $p \sim 2q^{3/2}$ (ignoring an overall $\pm$). One then sets $\partial S/\partial q = 2q^{3/2} + \partial S'/\partial q$ and solves

$$2q^{3/2} \partial S'/\partial q = \frac{a}{2}q + E - \frac{1}{2}(\partial S'/\partial q)^2$$

recursively for S'. The first reasonable approximation $\tilde{S}$ to S is

$$\tilde{S}(q, v) = \pm \left(\frac{4}{5}q^{5/2} + \frac{a}{2}q^{1/2} - vq^{-1/2} \right). \tag{2.6}$$

Since we are truncating S at the order shown, E is no longer constant and becomes the variable v at infinity. Clearly if more terms in the expansion of S were retained v would approximate E to higher order in $(t - t_0)$. By defining

$$u = \partial \tilde{S}/\partial v, \quad p = \partial \tilde{S}/\partial q, \tag{2.7}$$

we obtain the transition functions

$$q = u^{-2},$$
$$p = -2u^{-3} - \frac{a}{4}u - \frac{1}{2}vu^3. \tag{2.8}$$

Note that the $\pm$ in $\tilde{S}$ disappears. The points added to augment the manifold are $\{u = 0, v \in C\}$. One can also verify directly that $dq \wedge dp = du \wedge dv$. The Hamiltonian in the principal patch reads

$$\mathscr{H}(u, v) = v + \frac{a^2}{32}u^2 + \frac{a}{8}vu^4 + \frac{1}{8}v^2u^6, \tag{2.9}$$

and one can verify that

$$\dot{u} = \partial\mathscr{H}/\partial v = 1 + \mathcal{O}(u).$$

It might appear that the inverse of (2.8) is multivalued around $u = 0$, but this does not follow once one realizes that the domain of (2.8) and thus the patch covering $\{u = 0, v \in C\}$ is restricted to a tube around $u = 0$. Therefore we can find u recursively by solving

$$-2q/p = u\Big/\Big(1 + \frac{a}{8}u^4 + \frac{1}{4}vu^6\Big),$$

and obtain v by solving the quadratic equation $\mathscr{H}(u, v) = \mathscr{H}(p, q)$. The desired root is the one for which $v \to \infty$ as $u \to 0$ and $\mathscr{H}$ is fixed.

We now discuss the four steps required to construct M in more detail.

2.2.1. Hamilton–Jacobi expansion for a balance

Given a balance, weight the variables $\{p, q\}$ according to their leading exponents in (1.2) i.e., $q_i \sim t^{-f_i}$, $p_i \sim t^{-g_i}$. The principal balances are singled out in our discussion since more is known about them. In particular, by suitably partitioning the constants we can find the first few terms in the expansion for S by a variable change (appendix A) based on the series. It is at this point also where one has a precise but not necessarily unique criterion again based on the pole series as to which variable in a conjugate pair is p and which is q. With this done, we set $p = \partial S/\partial q$ and solve

$$\mathscr{H}\left(\frac{\partial S}{\partial q}, q\right) = E \tag{2.10}$$

iteratively. The terms in S will be ordered by their degree, l, with respect to t, i.e., $S_l \sim t^l$, (l is an integer),

$$S = S_{-r} + S_{-r+1} + S_{-r+2} + \cdots, \tag{2.11}$$

where $r > 0$. For a principal balance, the first term, S_{-r}, is frequently a monomial $q_1^{v_1}q_2^{v_2}\ldots$ $v_i \in Q$ but in general is a homogeneous function of q with no free constants, i.e., it has a precise homogeneous degree. The iteration is done by linearizing (2.10) about the piece of S known up to that point and inverting the resulting linear operator.

In the appendix it is shown for principal balances that free parameters only enter S through S_i with $i \geq 0$ (i.e., S_i is bounded as $q_j \to$ infinity). The diverging terms, $S_{i<0}$, in (2.11) are therefore finite sums of monomials with well defined coefficients. Free parameters only enter through the *free functions* in the kernel of the linearization of (2.10). (With one degree of freedom systems the one free constant is the energy which appears explicitly in the equation.)

To determine the free functions exactly would be equivalent to having the other integrals in involution and it is unreasonable to expect local analysis to yield this information directly. (Recall that from n

integrals in involution we can solve for $\partial S/\partial q_i$ as a function of $\{q\}$ and compute S by quadratures.) For the purposes of constructing transition functions, we can choose an arbitrary free function so long as the constant thereby introduced matches the constant introduced at the analogous order in the pole series. At a principal balance it should be remembered that the transition functions are just a symplectically consistent truncation of the pole series. For any balance, we expect to obtain $n - 1$ free constants from the free functions in the kernel, which together with E determine the new canonical momenta. It thus appears impossible to have a balance with fewer than n resonances for a Hamiltonian system.

The parallels between the Painlevé analysis of a system of differential equations and our Hamilton–Jacobi expansion should be noted. One first enumerates all "balances" by examining all pairs of monomials. Free constants or functions then enter through the kernel of a linearized operator.

There will be generating functions S which satisfy (2.10) and appear to give rise to lower balances with fractional exponents in time. These do not exist as true solutions of Hamilton's equations for $\mathcal{H}$ since the leading coefficients are all zero; however, they will correspond to acceptable lower balances for one of the Hamiltonians in involution with $\mathcal{H}$. (Two Hamilton–Jacobi equations are compatible if and only if the associated Hamiltonians are in involution.)

2.2.2. Transition functions

For any balance, the Hamilton–Jacobi equation for $\mathcal{H}(p, q)$ will generate a transition function, which is canonical, from (p, q) to the coordinate patch covering that balance. At a principal patch, we define for an n parameter truncation of S, $\tilde{S}(q, v_n, \ldots, v_{2n-1})$,

$$u = \partial \tilde{S}/\partial v_{2n-1}, \quad v_i = \partial \tilde{S}/\partial v_{n+i-1}, \quad i = 1, 2, \ldots, n-1,$$
$$v_{2n-1} = E. \tag{2.12}$$

Although E is one of the n parameters in $\tilde{S}$, it would be notationally confusing to use it as a variable on the coordinate patch since using (2.12) to express $\mathcal{H}$ in terms of u, $\{v\}$ results in a nontrivial polynomial which is linear in v_{2n-1} but contains the other coordinates as well, i.e., "E" is no longer a constant. The submanifold added to $M - U_0$ (i.e., the piece of M at "infinity") for a principal balance is just $u = 0$, $\{v\} \in C^{2n-1}$ in local coordinates; we now establish this.

The Hamilton–Jacobi expansion can always be continued to a sufficiently high order such that in u, $\{v\}$ variables Hamilton's equations read

$$\dot{u} = 1 + \mathcal{O}(u^m),$$
$$\dot{v}_i = \mathcal{O}(u^m) \tag{2.13}$$

for any integer $m > 0$. The function $\tilde{S}$ accomplishes in a canonical way the variable change from $\{p\}$ as a Laurent series with $2n - 1$ constants to $\{p\}$ as a function of $\{q\}$ and the n constants with largest ρ_j, $(v_{n-1}, \ldots, v_{2n-1}$ above). The variable, u, conjugate to E becomes identical to $t - t_0$ plus small corrections of order $(t - t_0)^{m+1}$. Similarly, all the other v_i approximate the pole constants with corrections that can be made of order $(t - t_0)^{m+1}$. The transition functions derived from $\tilde{S}$ must agree with more and more terms in the pole series when we substitute in a solution of (2.10) up to some high order. Hence $p, q(u, \{v\})$ are rational in u and polynomial in $\{v\}$. "Infinity" is just the surface $u = 0$.

For the lower balances, a $2n$ to $2n$ variable change is still required since these balances have to be adjoined as patches on M to the original $\{p, q\}$ phase space. However, only a submanifold of

codimension two or higher is actually added. Local variables can be chosen so that its equation is

$$u_1 = u_2 = \cdots u_j = 0$$

for a balance with $2n - j$ resonances. Furthermore, $u_j \neq 0$ corresponds to the next higher balance.

Let $v_1, \ldots, v_{2n-j}$ be the other local variables. The expansion for the generating function can be carried to an order such that $\{v\}$ agree to $\mathcal{O}(t - t_0)^{m+1}$ with the resonance constants in the series as before. Hence we have

$$\dot{u}_1 = 1 + \mathcal{O}(u_1^m),$$

$$\dot{v}_i = \mathcal{O}(u_1^m), \quad i = 1, \ldots, 2n - j.$$

However the equations for the remaining local variables $u_2, \ldots, u_j$ cannot be controlled by reference to the pole series. In the examples, we find equations of the form

$$\dot{u}_i = \mathcal{O}(u_1^{-\nu_i}), \quad \nu_i \in Q, \quad 0 < \nu_i < 1, \quad i = 2, \ldots, j.$$

The $u_{i \geq 2}$ are analogues of the Kovalevskaya resonances with negative exponents.

2.2.3. Flows at infinity

For the principal balance patch, the flows obviously extend through infinity. Furthermore $\mathcal{H}$ is polynomial in $u_1, \{v\}$. It is polynomial in $\{v\}$ since the transition functions are, and their poles in u_1 must cancel since $u_1 \sim t - t_0$ and $\mathcal{H}$ is invariant.

Since the $u_1, \{v\}$ differential equations are polynomial, analytic solutions exist around $u_1 = 0$. Hence the original pole series for $\{p, q\}$ converge to meromorphic functions in $(t - t_0)$. The most reasonable way to prove a singular series solution to a differential equation exists is to change coordinates to obtain a regular equation around infinity to which Picard iteration can be applied. Constructing an augmented manifold systematizes this procedure.

The lower balance flows have contact of order greater than one with the hypersurface at infinity $(M - C^{2n})$ (e.g. the hypersurface is $u_1 \sim u_2^3$ if $\nu_2 = 2/3$ in section 2.2.2 above). One may have to check on a case by case basis that the $\{u_{i \geq 2}\}$ equations do not branch around infinity, but the pole series for the corresponding lower balance are single valued by the Painlevé property.

2.2.4. Transition functions between added patches

The Hamiltonian equations of motion in the principal balance patch are well defined for all u, $\{v\} \in C^{2n}$ but we will only need them for u small. Clearly any singularity we find with $u \neq 0$ and some $v \rightarrow \infty$ should not be considered in this patch and can be ignored. The true singularities in u, v variables are those for which $v \rightarrow \infty$ as $u \rightarrow 0$ and these lie in a lower balance patch. In general we only have to use the principal balance patch in a tubular neighborhood of $u = 0$ whose radius tends to zero as $v \rightarrow \infty$. Fortunately it is precisely under those conditions that we can show from the differential equation (2.13) that the transition functions $p, q(u, v)$ are uniquely invertible.

The next set of lower balances will appear, attached to some principal balance, as singularities $u \rightarrow 0, v \rightarrow \infty$. A Painlevé analysis can be done on the $u, \{v\}$ differential equations to find pole series with the appropriate number of free constants. The associated Hamilton-Jacobi expansion can then be performed and transition functions determined. The process of enumerating all the singularities of Hamilton's equations in each patch defines a natural inclusion relation on the set of all balances.

2.3. *Examples*

The general discussion in the preceding section and the appendix will be illustrated with a series of examples of increasing complexity. The pole series are brought in to verify (A.1) and the pairing between resonance degrees (A.2).

2.3.1. *Jacobi elliptic functions*

The Hamiltonian is $2\mathcal{H} = p^2 - q^4 - aq^2$ and there are now two principal balances

$$q = \pm \frac{1}{(t - t_0)}(1 + \cdots),$$

$$p = \dot{q} \tag{2.14}$$

and two patches, $\pm$. As above we derive

$$\tilde{S} = \pm\left(\frac{1}{3}q^3 + \frac{a}{2}q - vq^{-1}\right)$$

and apply (2.11) to obtain the transition functions

$$q = \mp u_\pm^{-1},$$

$$p = \pm\left(u_\pm^{-2} + a/2 + u_\pm^2 v_\pm\right). \tag{2.15}$$

The two balances are distinct and we must add to $p, q \in C$ both sets $\{u_\pm = 0, v_\pm \in C\}$.

The transition functions are globally uniquely invertible wherever they are defined. Therefore we can take the patches covering infinity to be just $\{u, v\} \in C^2$. The Hamiltonian in u, v variables is just

$$\mathcal{H} = v\left(1 + \frac{1}{2}au^2\right) + \frac{1}{2}v^2u^4$$

for each balance, so again the u equation has the desired form, $\dot{u} = 1 + \mathcal{O}(u)$.

2.3.2. *Painlevé II*

The equation of motion is $\ddot{q} = 2q^3 + tq$. To conform with the definitions in section 1 we will make it an autonomous Hamiltonian system with 2 degrees of freedom. Let

$$2\mathcal{H} = p_1^2 - q_1^4 - q_2q_1^2 + 2p_2, \tag{2.16}$$

so that q_2 is the time, $q_1 = q$, and $p_1 = \dot{q}$.

There are two principal balances in direct analogy with the Jacobian elliptic functions,

$$q_1 = \pm \frac{1}{(t - t_0)}(1 + \cdots),$$

$$p_1 = \dot{q}_1,$$

$$q_2 = \text{const} + (t - t_0), \tag{2.17}$$

$$p_2 = \frac{-1}{t - t_0}\left(\frac{1}{2} + \cdots\right),$$

and no lower balances.

The explicit calculation of pole series is facilitated by using the nonautonomous form of $\mathcal{H}$. Set $q_2 = c_1 + t$, $p_2 = c_2 + \int_0^t q_1^2 \, dt$ and assume q_1, p_1 blow up at $t = 0$. Then $2\mathcal{H} = p_1^2 - q_1^4 - (t + c_1)q_1^2$ admits the symmetry $(q_1 p_1) \rightarrow -(q_1 p_1)$ so we need do the calculation only for one choice of signs. The series is then

$$q_1 = t^{-1} - \frac{c_1}{6}t - \frac{1}{4}t^2 + c_3 t^3 + \cdots .$$

One can verify, (cf. A.1),

$$\sum dp_i \wedge dq_i = dt_0 \wedge dE(c_1, c_2, c_3) + dc_2 \wedge dc_1,$$

and $E \equiv -5c_3 + c_2 + \frac{7}{32}c_1^2$ (i.e., E is $\mathcal{H}$ evaluated for the pole series). The resonance degrees of t_0 and c_3 satisfy $\rho_0 + \rho_3 = 3 = g_1 + f_1$ and for c_1 and c_2 we have $\rho_1 + \rho_2 = 1 = g_2 + f_2$ in conformity with (A.2).

The Hamilton–Jacobi equation reads

$$2E = \left(\frac{\partial S}{\partial q_1} \right)^2 + 2\frac{\partial S}{\partial q_2} - q_1^4 - q_2 q_1^2. \tag{2.18}$$

If we assign q_1 a weight 1 and q_2 a weight 0, then the largest monomial in S when differentiated and substituted must cancel q_1^4. Hence to leading order $S \sim \pm \frac{1}{3}q_1^3$. This result would also follow by observing from the series that $p_1 \simeq q_1^2$. If we now expand (2.18) by setting $S = \pm \frac{1}{3}q_1^3 + S'$, it becomes

$$\pm 2q_1^2 \frac{\partial S'}{\partial q_1} + 2\frac{\partial S'}{\partial q_2} = q_2 q_1^2 + E - \left(\frac{\partial S'}{\partial q_1} \right)^2. \tag{2.19}$$

Only the first term in the linear operator on the left has to be retained since the second is of lower weight. An iteration can now be done, and the kernel of $\partial/\partial q_1$ acting on S' is clearly an arbitrary function of q_2. It is expedient, however, to simply solve (2.18) for $\partial S/\partial q_1$ and iterate as in the elliptic case, viz.,

$$S = \pm \int q_1^2 \left(1 + \frac{1}{2}q_2 q_1^{-1} - \left(\frac{\partial S}{\partial q_2} + \frac{1}{8}q_2^2 \right) q_1^{-4} + E q_1^{-4} + \cdots \right) dq_1 + f(q_2, a),$$

where a second free constant a, has been introduced through the free function f in the kernel of $\partial/\partial q_1$. We have immediately

$$S = \pm \left(\frac{1}{3}q_1^3 + \frac{1}{2}q_2 q_1 \right) + \mathcal{O}(1),$$

which is then to be used to compute $\partial S/\partial q_2$ in the integral to continue the iteration. We therefore obtain

$$\tilde{S} = \pm \left(\frac{1}{3}q_1^3 + \frac{1}{2}q_2 q_1 \right) - \frac{1}{2} \ln(q_1) - f \pm \left(\frac{\partial f}{\partial q_2} - E + \frac{1}{8}q_2^2 \right) q_1^{-1} + \mathcal{O}(q_1^{-2}). \tag{2.20}$$

Now set $f = -aq_2 - \frac{1}{24}q_2^3$ and compute the symplectic variable change from

$$u = \frac{\partial S}{\partial E}, \quad b = \frac{\partial S}{\partial a}.$$

One obtains (suppressing the $\pm$ subscripts on all variables)

$$u = \mp q_1^{-1},$$

$$b = q_2 \mp q_1^{-1},$$

$$p_1 = \pm(q_1^2 + q_2) - \tfrac{1}{2}q_1^{-1} \pm (a + E)q_1^{-2},$$

$$p_2 = \pm q_1 + a + \tfrac{1}{8}q_2^2. \qquad (2.21)$$

While (2.21) satisfies the technical requirements for the variable change to the principal balance patch, more attractive transition functions are obtained by a further canonical variable change to (u, v) coordinates,

$$v_1 = a + \tfrac{1}{8}q_2^2,$$

$$v_2 = b \pm q_1^{-1} = b - u = q_2,$$

$$v_3 = a + E,$$

for which

$$\sum dp_i \wedge dq_i = dE \wedge du + da \wedge db$$

$$= dv_3 \wedge du + dv_1 \wedge dv_2.$$

Finally the transition functions are, (suppressing $\pm$ subscripts on u, v)

$$q_1 = \mp u^{-1},$$

$$p_1 = \pm(u^{-2} + v_2/2 + u/2 + u^2 v_3),$$

$$q_2 = v_2, \qquad (2.22)$$

$$p_2 = -u^{-1}/2 + v_1.$$

We can again use for the two principal balance patches, $\{u_+, v_+\} = U_+ = C^4$, since the transformation (2.22) is globally invertible wherever it is defined. Comparable transition functions were obtained by Okamoto [13] after a lengthy sequence of local algebraic transformations. The Hamiltonian is identical in the two new patches and reads

$$\mathscr{H} = \tfrac{1}{8}v_2^2 + v_1 + v_3 + \tfrac{1}{4}v_2 u + \tfrac{1}{2}u^2 v_3 v_2 + \tfrac{1}{8}u^2 + \tfrac{1}{2}u^3 v_3 + \tfrac{1}{2}u^4 v_3^2.$$

2.3.3. Integrable "Hénon–Heiles"
The Hamiltonian reads

$$\mathscr{H} = \tfrac{1}{2}(p_1^2 + p_2^2) + q_1^2 q_2 + 2q_2^3 + \tfrac{1}{2}(q_1^2 + q_2^2). \qquad (2.23)$$

The Painlevé test for (2.23) was done in [3] and Greene found the second integral in involution. We will omit the quadratic terms in q_i in most of what follows to simplify the algebra. Also the transition functions are taken only far enough make $\dot{v}_i \sim$ const. rather than $\mathcal{O}(u)$ which is adequate to show in this

example that the transition functions are invertible and separate the flows at infinity. Our discussion is broken into several subsections for clarity.

a. *Series and resonances*

There is a single principal balance with leading exponents $f_i = (1, 2)$, $g_i = (2, 3)$ and series

$$q_1 = c_1 t^{-1} + \tfrac{1}{12}(5c_1 + c_1^3)t + c_2 t^2 - \left(\tfrac{1}{72}c_1^5 + \tfrac{11}{144}c_1^3 + \tfrac{61}{720}c_1\right)t^3$$

$$- \tfrac{5}{6}c_2(1 + c_1^2)t^4 - \tfrac{1}{9}(c_1 c_3 + P_7(c_1))t^5 + \cdots, \qquad (2.24)$$

$$q_2 = -t^{-2} + \tfrac{1}{12}(c_1^2 - 1) + \left(\tfrac{1}{48}c_1^4 + \tfrac{1}{12}c_1^2 - \tfrac{1}{240}\right)t^2 + \tfrac{1}{3}c_1 c_2 t^3 + c_3 t^4 + \cdots.$$

For simplicity we have written t for $t - t_0$. When $(q_1^2 + q_2^2)$ is omitted only the highest powers of c_1 remain in each of the coefficients. (P_7 is a seventh degree polynomial in c_1.) By inspection, resonances occur for $\rho = -1, 0, 3$, and 6.

There are two lower balances with two free constants. The series begin as

$$q_1 = \pm 6 i t^{-2}(1 + \cdots),$$
$$q_2 = -3t^{-2}(1 + \cdots) \qquad (2.25)$$

and there are resonances for $\rho = -1, 6, 8$.

The resonance constants in the principal balance series satisfy the relations in the appendix. The energy becomes

$$E = 14c_3 + \tfrac{35}{432}c_1^6$$

and

$$\sum_i dp_i \wedge dq_i = dt_0 \wedge dE + 3 dc_2 \wedge dc_1. \qquad (2.26)$$

Furthermore the resonance degrees obey ($\rho_{1,2}$ correspond to $c_{1,2}$)

$$\rho_t + \rho_E = 5 = f_2 + g_2,$$
$$\rho_1 + \rho_2 = 3 = f_1 + g_1.$$

b. *Transition functions for the principal patch*

The Hamilton–Jacobi equation with $x = q_1$, $y = -q_2$ and the quadratic terms omitted reads

$$E = \frac{1}{2}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right) - x^2 y - 2y^3. \qquad (2.27)$$

The expansion begins as

$$S = \tfrac{4}{5}y^{5/2} + \tfrac{1}{2}x^2 y^{1/2} + \cdots$$

and the linear operator $\mathscr{L}$ acting on S that must be inverted at every iteration is

$$\mathscr{L}(S) = 2y^{3/2}\partial S/\partial y + xy^{1/2}\partial S/\partial x.$$

Both terms are of the same order.

The kernel of $\mathscr{L}$ is any function of the form $f(xy^{-1/2}, a)$ and enters S at order t^0. (It is not coincidental that $z = xy^{-1/2}$ picks out the leading free constant in the pole series, see below.) There are no inhomogeneous terms in the linearization of (2.27) at this order and no logs enter S.

Suppressing an overall $\pm$,

$$\tilde{S} = \tfrac{4}{5}y^{5/2} + \tfrac{1}{2}x^2y^{1/2} - \tfrac{1}{32}x^4y^{-3/2} + f$$

$$-Ey^{-1/2} + \tfrac{5}{256}x^6y^{-7/2} - \tfrac{1}{8}x^3y^{-5/2}\frac{\partial f}{\partial z} + \mathcal{O}(t^3).$$ (2.28)

If we include terms only up to $Ey^{-1/2}$ in S, set $f = -v_2xy^{-1/2}$, $v_3 = E$, $u = \partial\tilde{S}/\partial v_3$, and $v_1 = \partial S/\partial v_2$, we obtain the transition functions

$$q_1 = v_1 u^{-1},$$

$$q_2 = -u^{-2},$$

$$p_1 = -v_1 u^{-2} + \tfrac{1}{8}v_1^3 + v_2 u,$$

$$p_2 = 2u^{-3} + \tfrac{1}{4}v_1^2 u^{-1} + \tfrac{3}{64}v_1^4 u + \tfrac{1}{2}v_1 v_2 u^2 + \tfrac{1}{2}v_3 u^3,$$ (2.29)

and as usual $\Sigma\, dq_i \wedge dp_i = du \wedge dv_3 + dv_1 \wedge dv_2$.

By truncating $\tilde{S}$ where we did, an $\mathcal{O}(t)$ term was omitted which is comparable to $Ey^{-1/2}$. As a consequence $\dot{v}_2 = \mathcal{O}(1)$ at $u = 0$. If we compare $\dot{q}_i$ from (2.24) with p from (2.29), then the first occurrence of $v_{2,3}$ matches that of $c_{2,3}$ to within numerical factors. Also $v_1 \to c_1$ as $u \to 0$.

It is of some interest in this example to find the exact kernel of $\mathscr{L}$, f in (2.28). By eliminating p_i between (2.23) (with $q_1^2 + q_2^2$ omitted) and the second integral

$$G = 4p_1 p_2 q_1 - 4p_1^2 q_2 + 4q_1^2 q_2^2 + q_1^4,$$ (2.30)

one finds

$$\partial f/\partial(xy^{-1/2}) = -\tfrac{1}{2}(G - \tfrac{1}{16}x^8 y^{-4} + 2Ex^2y^{-1})^{1/2}.$$ (2.31)

We can also express G in terms of the pole constants in (2.24),

$$G = 36c_2^2 - 2c_1^2 E + \tfrac{1}{16}c_1^8.$$ (2.32)

To evaluate f near infinity we observe from (2.24) that $x^2/y = c_1^2$ and substitute (2.32) into (2.31) with the result

$$\partial f/\partial(xy^{-1/2}) = -3c_2.$$

This precisely agrees with our choice of f below (2.28) which was made on the grounds of simplicity (the factor of 3 can be checked by comparing the expression for $\Sigma\, dq_i \wedge dp_i$ in terms of u, v against (2.26)).

Without appealing to "simplicity" an adequate expression for f can always be found by comparing the series with the transition functions and demanding v_i appear as c_i.

The domain of (2.29) consists of $v_i \in C^3$ and u restricted to a tube around the hypersurface at infinity, $u = 0$. The Hamiltonian, with $(q_1^2 + q_2^2)$ omitted, in local coordinates reads

$$\mathcal{H} = v_3 + \tfrac{5}{256}v_1^6 + \tfrac{1}{4}v_1^3 v_2 u + \left(\tfrac{1}{2}v_2^2 + \tfrac{9}{8192}v_1^8 + \tfrac{1}{8}v_1^2 v_3\right)u^2$$

$$+ \tfrac{3}{128}v_1^5 v_2 u^3 + \left(\tfrac{1}{8}v_1^2 v_2^2 + \tfrac{3}{128}v_1^4 v_3\right)u^4 + \tfrac{1}{4}v_1 v_2 v_3 u^5 + \tfrac{1}{8}v_3^2 u^6. \tag{2.33}$$

The only singularities of Hamilton's equations for (2.33) as $u \to 0$ are $v_1 \sim u^{-1}$, $v_2 \sim u^{-4}$, and $v_3 \sim u^{-6}$ which must represent the lowest balances. These powers can be obtained by demanding that (2.29) match (2.25). It may be explicitly verified that (2.29) is invertible on a sufficiently large tube around $u = 0$ to encompass the lowest balances.

c. *Lower balance transition functions*

We now consider how to add a codimension 2 set to $M - U_0$ corresponding to the lowest balances (2.25). This requires that we solve the Hamilton–Jacobi equation (2.27) for $x, y \to \infty$ and x/y finite. Evidently all terms on the righthand side of (2.27) are of the same order so the leading term in S is of the form $y^{5/2}a(x^2/y^2)$, where a satisfies (with $z = x^2/y^2, ' = d/dz$)

$$4 + 2z = (4z + 4z^2)a'^2 - 10zaa' + 25a^2/4. \tag{2.34}$$

Since we will want to examine the crossover from the principal balance to (2.25), we have written the argument of a as x^2/y^2 rather than y^2/x^2. Hence we seek a solution to (2.34) which is analytic out to $z = -4$ and satisfies $a(0) = 4/5$ from (2.28). We will assume this can be done; to prove it for this example one would exploit the separability of (2.18).

We know from the resonances calculated for (2.25) that free constants enter S at order $y^{-1/2}$ and $y^{-3/2}$ i.e., at orders t and t^3 relative to a leading order of t^{-5}. (Recall that the transition functions we derive from $\tilde{S}$ must match the series.) The first three terms in (2.28) together with a piece of f (cf. (2.31)) all fall into a.

We next linearize (2.27) about $S = y^{5/2}a$ and observe that the next term is of the form $y^{-1/2}b(z)$ where $b(0) = -E$ in order to agree with (2.28). Since $S = y^{5/2}a$ satisfies (2.27) exactly with $E = 0$, there are no correction terms until E appears. The next free constant enters S in the form $y^{-3/2}c(z, v_2)$. The precise forms of b and c are not needed to match the pole series. It suffices as before to take just the first nonzero term around $z = 0$,

$$\tilde{S} = y^{5/2}a(z) - Ey^{-1/2} - v_2 y^{3/2}/x^3. \tag{2.35}$$

(In this example we did not actually compute (2.25) out to order 8, which could be done, but rather expanded (2.31).)

The remaining variables in the patch at infinity are defined by

$$u_i = \partial\tilde{S}/\partial v_i,$$

$$v_1 = E.$$

Then,

$$q_1 = u_2^{-1/3}/u_1,$$

$$q_2 = -u_1^{-2}. \tag{2.36}$$

Obviously when $u_2 \neq 0$ (2.36) again describes the principal balance. The lowest balance occurs for $u_1 = u_2 = 0, v_1, v_2 \in C^2$.

The Hamiltonian in $\{u, v\}$ coordinates is complicated but the leading dependence on v_i is simple and permits us to derive for $q_2/q_1 \sim \mathcal{O}(1)$,

$$\dot{u}_1 = 1,$$

$$\dot{u}_2 = u_1 u_2^{1/3}. \tag{2.37}$$

Numerical constants have been ignored in (2.37). Therefore the lowest balance flows pass infinity according to $u_2 \sim t^3, u_1 \sim t$. In the vicinity of the lowest balance, the hypersurface at infinity in local coordinates is just $u_1^3 \sim u_2$ (i.e., $q_2/q_1 \sim$ const.). The flow (2.37) is therefore tangent.

d. *Balances and resonances for G*

It is an interesting exercise to repeat the preceeding discussion for the integral G, (2.30) which is in involution with $\mathcal{H}$, (2.23). Actually it is more illuminating to do so taking account of the separability of this system as we do in the next section. At this point we will merely make an observation about the lower balances to illustrate a remark made at the end of section 2.2.1 above.

The Hamilton equations for $G(p, q)$ appear to have a balance in which

$$q_i \sim t_2^{-2/3}, \quad p_i \sim t_2^{-1},$$

where t_2 is the flow generated by G (and we use t_1 in place of t for the $\mathcal{H}$ flow). However, although the exponents balance out, there is no solution other than zero for the coefficients.

Actually this "phantom" balance should not be a surprise since in (2.35) $v_2 \sim G$ and $u_2 \sim t_2$. Hence if (2.37) is solved for $u_1(u_2)$ and substituted into (2.36), a $-2/3$ exponent is found. Similarly from (2.37), $p_i \sim t_1^{-3} \rightarrow t_2^{-1}$. But separability for Hénon–Heiles implies there are no fractional powers, hence the coefficients vanish.

It will be useful later on to remark that integration of the equations for (2.32) is another way to complete the principal balance submanifold at infinity and obtain all of $M - U_0$. The flow found in this way is nothing but the full t_2 flow projected onto infinity. From (2.26) we can write Hamilton's equations for (2.32) and find

$$c_1 \sim t_2^{-1/3}, \quad c_2 \sim t_2^{-4/3}$$

and

$$\partial t_1/\partial t_2 = \partial G/\partial E, \quad t_1 \sim t_2^{1/3}.$$

We will explain in the next section why both $\mathcal{H}$ and G are homogeneous when $\{p, q\}$ are weighted according to the lowest balance exponents of $\mathcal{H}$. Yoshida, [15], has also explored the implications of

homogeneity in Painlevé systems. Certain of his arguments assume that if one can assign $\{p, q\}$ exponents which render an integral homogeneous, then the same exponents lead to a balance. This is false in separable systems except for the "lowest" Hamiltonian of the system (i.e., $\mathcal{H}$) and the "phantom" balance for G illustrates what goes wrong; the leading coefficients vanish.

There are actually two additional lower balances for G which do not resemble (2.25) at all (see section 3.5.4).

3. Separable systems

3.1. Definition

Definition 1. The Hamiltonian system (1.1) is *hyperelliptically separable* (h.s.) if there is a transformation

$$q = q(\xi)$$

such that
a. q_i is a symmetric polynomial function of the $\{\xi_j\}$;
b. under this transformation, the Hamilton–Jacobi equation separates and the resulting action becomes

$$S = \sum_{i=1}^{n} S_i(\xi_i; h_1, \ldots, h_n),$$

$$S_i = \int^{\xi_i} \eta \, d\xi,$$

where

$$\eta^2 = \xi^d + \alpha_1 \xi^{d-1} + \cdots + \alpha_d$$

is a polynomial of degree $d \geq 2n + 1$;
c. the free parameter, h_i, is just α_{j_i} where $[(d+4)/2] \leq j_{i_1} < j_{i_2} < \cdots j_{i_n};$*
d. the induced transformation on phase space is

$$q = q(\xi),$$

$$p = p(\xi, \eta) = \eta \cdot \frac{\partial \xi}{\partial q},$$

where $\eta = (\eta(\xi_1), \ldots, \eta(\xi_n)) = \partial S / \partial \xi$. Over the finite part of phase space, $(q, p) \in C^{2n}$, we view this transformation as a map

$$\Phi: \tilde{M}' \to C^{2n},$$

where $\tilde{M}' = $ unordered n-tuples $\{(\xi_1, \eta_1), \ldots, (\xi_n, \eta_n)\}$. (Later, when we augment C^{2n}, we will let (ξ, η) become infinite and denote the corresponding augmentation by $\tilde{M}$.) Also we will denote the level set of

---

*[ ] denotes "the integer part of".

fixed h_i by $\tilde{M}'_h$. Φ is required to satisfy the following nondegeneracy conditions (which are explained in the subsequent remarks):

i. Off of a codimension two subset, Σ, of C^{2n}, Φ is a finite, unramified covering. Also $\Phi(\Phi^{-1}(\Sigma) \cap \tilde{M}'_h)$ has codimension two in Φ ($\tilde{M}'_h$);

ii. near $\xi_1 = \infty$, $q(\xi)$ is an invertible transformation almost everywhere;

iii. on an open dense subset of constant values $c \in C^n$, the determinant

$$\left| \frac{\partial^2 S}{\partial \xi_i \partial h_j} \right| = \left| \frac{\partial \eta_i}{\partial h_j} \right| \neq 0$$

along the level set $\tilde{M}'_c$ but away from $\Phi^{-1}(\Sigma)$ (this non-vanishing also applies to locations at which $\xi_i = \infty$, where the determinant is evaluated in local coordinates at infinity – see remark 4).

Remark 1. In part d(i) of the definition, "finite, unramified cover" means that there is a finite number, m, such that for each (q, p) in the range of Φ there are *exactly* m distinct preimages in $\tilde{M}'$ which map to that point. This condition is required to insure that the integrals of the separable system are all analytic (see theorem 3b).

Remark 2. Condition d(ii) is a technical assumption. To give it a more precise statement we expand each q_i as a polynomial in ξ_1 with coefficients involving $\xi_2, \ldots, \xi_n$:

$$q_i = q_i^0(\xi_2, \ldots, \xi_n)\xi_1^{l_i} + q_i^1(\xi_2, \ldots, \xi_n)\xi_1^{l_i - 1} + \cdots$$

and set $Q_i^0 = q_i^0 \cdot \xi_1^{l_i}$; then

$$\det\left(\frac{\partial Q^0}{\partial \xi} \right) = \xi_1^{(\Sigma l_i) - 1} \det \begin{bmatrix} l_1 q_1^0 & \dfrac{\partial q_1^0}{\partial \xi_2} & \cdots & \dfrac{\partial q_1^0}{\partial \xi_n} \\ \vdots & \vdots & & \vdots \\ l_n q_n^0 & \dfrac{\partial q_n^0}{\partial \xi_2} & \cdots & \dfrac{\partial q_1^0}{\partial \xi_n} \end{bmatrix}.$$

Condition d(ii) is equivalent to the determinant on the right not being identically zero.

For these expressions to make sense we must have $l_i > 0$. If $l_i = 0$, then q_i is independent of ξ_1. But q_i is symmetric in ξ and therefore this would imply that q_i is constant. This violates d(i) since Φ could not then be a finite map.

Remark 3. Condition d(iii) will insure that the series solution of a separable system will be Laurent. Without it one could not preclude the possibility of the leading exponents being non-integral. In fact they can be fractional.

Remark 4. The "points at ∞" in (ξ, η)-space have local coordinates given by the variable change

$$\xi = \frac{1}{\lambda}, \quad \eta = \frac{p}{\lambda^{d/2}} \quad \text{when } d \text{ is even}$$

and

$$\xi = \frac{1}{\lambda^2}, \quad \eta = \frac{p}{\lambda^d} \quad \text{when } d \text{ is odd.}$$

In the first case there are two points at ∞, $(\lambda, p) = (0, \pm 1)$ while in the second there is just $(\lambda, p) = (0, 1)$, i.e., infinity is a branch point.

Remark 5. Part c is more restrictive than need be. In general α_j, $j \geq [(d + 4)/2]$, could be more general functions of $h_1, \ldots, h_n$; i.e., we just need that $\alpha(h)$ embed an open dense subset of $h \in C^n$ as a complex submanifold of $C^{[(d-1)/2]}$. (We must require that α_j, for $j < [(d + 4)/2]$, be independent of h in order that $\partial S / \partial h$ be finite at infinity as d(iii) requires.) The arguments of this section can be extended to the more general situation by setting $\alpha_{j_1}, \ldots, \alpha_{j_n}$ equal to the first n coefficients of η^2 which constitute a functionally independent set as functions of h.

3.2. *Examples*

In this subsection we present two hierarchies of examples which will serve to illustrate the preceding definitions and which will also be used throughout the remainder of this paper to motivate our results.

(i) *The Gelfand-Dikii (G.D.) hierarchy*

(This also corresponds to a special class of higher stationary solutions for the Korteweg-deVries hierarchy [16].) This hierarchy is indexed by the number of degrees of freedom, n. It is completely specified by saying that for n, the polynomial η^2 in definition 1 is given by

$$\eta^2 = \xi^{2n+1} + h_1 \xi^{n-1} + \cdots + h_n \tag{3.1a}$$

and for the variable transformation, $q = q(\xi)$, one simply sets

$$q_i = \sigma_i(\xi) \tag{3.1b}$$

where σ_i is the ith symmetric polynomial in the variables $(\xi_1, \ldots, \xi_n)$. This is the simplest example in that the degree of η^2 is minimal: $d = 2n + 1$. One could modify this hierarchy by allowing the coefficients of $\xi^{2n}, \ldots, \xi^n$ in (2.1a) to be non-zero but fixed (i.e. independent of parameters h_i). This just causes the integrals of motion to be not "homogeneous" in a sense that will be specified later.

As an example, consider $n = 2$. The corresponding system (1.1) can be given by the Hamiltonian

$$\mathcal{H}_1 = -q_1 p_2^2 - 2p_2 p_1 + 3q_2 q_1^2 - q_1^4 - q_2^2 \tag{3.2a}$$

corresponding to h_1, and another integral of the motion is

$$\mathcal{H}_2 = 2p_1 p_2 q_1 + q_1^2 p_2^2 + p_1^2 - q_2 p_2^2 + q_2 q_1^3 - 2q_2^2 q_1 \tag{3.2b}$$

corresponding to h_2. Equally well, any linear combination of $\mathcal{H}_1$ and $\mathcal{H}_2$ could be taken to be $\mathcal{H}$ in defining (1.1). In any case the separating change of variables is

$$q_1 = \xi_1 + \xi_2, \quad q_2 = \xi_1 \xi_2.$$

(ii) *The Hénon–Heiles* (*H.H.*) *hierarchy*

The nth level in this hierarchy is specified by

$$\eta^2 = -\tfrac{1}{2}\xi^{4n} + h_1\xi^{2n-2} + h_2\xi^{2n-4} + \cdots + h_n \tag{3.3a}$$

and the variable transformation is

$$q_k = \tfrac{1}{2}\sigma_{k-1}\left(\xi_1^2,\ldots,\xi_n^2\right)$$

$$= \tfrac{1}{2} \sum_{i_1 < i_2 < \cdots i_k} \xi_{i_1}^2 \cdots \xi_{i_k}^2, \quad 2 \leq k \leq n, \tag{3.3b}$$

$$q_1 = \mathrm{i}\xi_1 \cdots \xi_n.$$

As an example, again consider $n = 2$. The Hamiltonian can be given by

$$\mathcal{H}_1 = p_1^2 + p_2^2 + 2q_1^2 q_2 + 4q_2^3 \tag{3.4a}$$

corresponding to h_1, with another integral

$$\mathcal{H}_2 = 2p_1 p_2 q_1 - 2p_1^2 q_2 + 2q_1^2 q_2^2 + \tfrac{1}{2}q_1^4 \tag{3.4b}$$

that corresponds to h_2. These were denoted by $2\mathcal{H}_1$ and $G/2$ in section 2.3.4.

3.3. *The Abel map and the linearization of flows*

Here we briefly review some standard results from the theory of Riemann surfaces in the context of our problem. For further details the reader is referred to [18, 19]. Given the separated action (definition 1b)

$$S = \sum_{j=1}^{n} \int^{\xi_j} \eta \, d\xi,$$

the linearized flows are implicitly given by

$$t_i = \frac{\partial S}{\partial h_i} = \sum_{j=1}^{n} \int^{\xi_j} \frac{\partial \eta}{\partial h_i} \, d\xi. \tag{3.5a}$$

In the Gelfand–Dikii example this is just

$$t_i = \tfrac{1}{2} \sum_{j=1}^{n} \int^{\xi_j} \frac{\xi^{n-i} \, d\xi}{\sqrt{\xi^{2n+1} + h_1\xi^{n-1} + \cdots + h_n}}, \tag{3.5b}$$

while for Hénon–Heiles this is

$$t_i = \tfrac{1}{2} \sum_{j=1}^{n} \int^{\xi_j} \frac{\xi^{2n-2i} \, d\xi}{\sqrt{-1/2\xi^{4n} + h_1\xi^{2n-2} + \cdots + h_n}}. \tag{3.5c}$$

If these flows are assembled in a vector $t = (t_1, \ldots, t_n)^{\mathrm{T}}$, we have

$$\sum_{i=1}^{n} \int^{\xi_i} \frac{\partial \eta}{\partial h} \, d\xi = \frac{\partial S}{\partial h} = t. \tag{3.6}$$

We define, for a fixed value of h,

$$\mathcal{R}_h = \left\{ (\xi, \eta) \big| \eta^2 = \xi^d + \alpha_1 \xi^{d-1} + \cdots + \alpha_d \right\} \cup \{\text{points at } \infty\}, \tag{3.7}$$

which is a hyperelliptic Riemann surface of genus $g = d/2 - 1$ (resp. $(d-1)/2$) if d is even (resp. odd). (We will often delete the subscript h and just write $\mathcal{R}$.) Eq. (3.6) gives a map

$$\mathcal{R}^{(n)} \to C^n,$$

where $\mathcal{R}^{(n)}$ is the nth symmetric product of $\mathcal{R}$ with itself. This is not quite well-defined because the integrals on the LHS have $2g$ independent periods.

When $n = g$ and $|\partial^2 S / \partial q \, \partial h| \neq 0$ the set of all such periods forms a maximal lattice $\Lambda \subset C^g$, and C^g/Λ is a complex g-dimensional torus. We have the following classical theorems [18, 19].

Theorem 1. C^g/Λ is an abelian variety (i.e. it can be analytically embedded into some sufficiently high dimensional complex projective space) called the *Jacobian of* $\mathcal{R}$ and denoted $\mathscr{J}(\mathcal{R})$.

Theorem 2. The map (3.6) induces the *Abel map*

$$
\begin{aligned}
&A: \mathcal{R}^{(g)} \to \mathscr{J}(\mathcal{R}), \\
&\left((\xi_1, \eta_1), \ldots, (\xi_g, \eta_g) \right) \to t \ (\text{mod periods } \Lambda).
\end{aligned} \tag{3.8}
$$

A, restricted to points where $(\xi_i, \eta_i) \neq (\xi_j, -\eta_j)$ for $i \neq j$, is $1:1$.

When $n < g$ we can define an extension of (3.6)

$$\sum_{i=1}^{n} \int^{\xi_i} \frac{\partial \eta}{\partial \beta} \, d\xi = \frac{\partial S}{\partial \beta} = (t_1', \ldots, t_g'), \tag{3.9}$$

where $\beta = (\alpha_{[(d+4)/2]}, \ldots, \alpha_d)$ (see definition 1). This gives a map

$$A_{(n)}: \mathcal{R}^{(n)} \to C^g.$$

Then projection $\beta \to h$, which just freezes those $\alpha_i \neq h_k$ for some k, induces a projection $\Pi: C^g \to C^n$. The original map (3.6) is just the composition $\Pi \circ A_{(n)}$.

Since the Abel map is invertible off of the codimension one subvariety $\{(\xi_i = \xi_j, \eta_i = -\eta_j)$ for some $i \neq j\}$; it follows that we can, in principle, solve for the ξ_i as functions of t. What we can do in practice is to write down differential equations for this dependence. Differentiating (3.6) w.r.t. t we have

$$\sum_{i=1}^{n} \frac{\partial \eta}{\partial h}(\xi_i) \cdot \frac{\partial \xi_i}{\partial t} = I. \tag{3.10}$$

Inverting, this gives

$$\frac{\partial \xi}{\partial t} = \left(\frac{\partial \eta}{\partial h}(\xi) \right)^{-1}. \tag{3.11}$$

For our special examples we can be more explicit. For the G.D. hierarchy (3 ^) the ξ-odes, (3.11), for the $\mathcal{H}_{k+1}$ flow are

$$\frac{\partial \xi_l}{\partial t_{k+1}} = \frac{(-1)^k 2\eta_l \sigma_k^{(l)}}{\prod_{j \neq l}(\xi_l - \xi_j)} \tag{3.12}$$

where $\sigma_k^{(l)} = k$th elementary symmetric polynomial in $(\xi_1, \ldots, \xi_{l-1}, \xi_{l+1}, \ldots, \xi_n)$. When $n = 2$ (see (3.2)) these equations are

$$\begin{pmatrix} \xi_{1t_2} & \xi_{1t_1} \\ \xi_{2t_2} & \xi_{2t_1} \end{pmatrix} = \frac{2}{\xi_2 - \xi_1} \begin{pmatrix} \xi_2 \eta_1 & -\eta_1 \\ -\xi_1 \eta_2 & \eta_2 \end{pmatrix}. \tag{3.12a}$$

For the H.H. hierarchy (3.3) the ξ-odes for $\mathcal{H}_{k+1}$ are

$$\frac{\partial \xi_l}{\partial t_{k+1}} = \frac{(-1)^k 2\eta_l \tilde{\sigma}_k^{(l)}}{\prod_{j \neq l}(\xi_l^2 - \xi_j^2)}, \tag{3.13}$$

where $\tilde{\sigma}_k^{(l)} = k$th elementary symmetric polynomial in $(\xi_1^2, \ldots, \xi_{l-1}^2, \xi_{l+1}^2, \ldots, \xi_n^2)$. When $n = 2$ (3.4) these are

$$\begin{pmatrix} \xi_{1t_2} & \xi_{1t_1} \\ \xi_{2t_2} & \xi_{2t_1} \end{pmatrix} = \frac{2}{\xi_2^2 - \xi_1^2} \begin{pmatrix} \xi_2^2 \eta_1 & -\eta_1 \\ -\xi_1^2 \eta_2 & \eta_2 \end{pmatrix}. \tag{3.13a}$$

If we set $p_{\xi_i} = \eta_i$, then $dS = \sum \eta_i \, d\xi_i$, and

$$(q, p) \rightarrow (\xi, p_\xi)$$

is a canonical variable change. Explicitly

$$p_{\xi_i} = \frac{\partial S}{\partial \xi_i} = \frac{\partial S}{\partial q_j} \cdot \frac{\partial q_j}{\partial \xi_i} = \frac{\partial q_j}{\partial \xi_i} p_j.$$

Thus,

$$p_i = \frac{\partial \xi_j}{\partial q_i} \cdot p_{\xi_j}. \tag{3.14}$$

Note, in particular, that if the q_i equal the symmetric functions σ_i, then p_i as a function on $\mathcal{R}^{(n)}$ is singular if and only if $\xi_j = \infty$ or $(\xi_i, \eta_i) = (\xi_j, -\eta_j)$ for some $i \neq j$.

3.4. The structure of balances

We will now characterize the structure of balances for a (h.s.) system.

Theorem 3. For a system, (1.1), which satisfies definition 1, the following are true:

(a) The Augmentation

The minimal augmentation, M, of the phase space C^{2n} can be given in terms of the separating variables (ξ_i, η_i) and h_k; a preliminary augmentation is given by

$$\tilde{M} = \{(h_1, \ldots, h_n; (\xi_1, \eta_1), \ldots, (\xi_n, \eta_n)) | h_i \in C; (\xi_i, \eta_i) \in \mathcal{R}\}.$$

We let $\tilde{M}_h$ denote the level set for fixed h_i.

(i) If $n = g$, the genus of $\mathcal{R}$, then $\tilde{M}$ is an augmentation of C^{2n}.

(ii) If $n < g$, then $\tilde{M}$ is a finite cover of an augmentation.

In either case, M, the minimal augmentation, is the image of $\tilde{M}$ under the Abel map $A_{(n)}$.

(b) Integrals

The parameters h_i can be expressed as global analytic functions on M. When restricted to C^{2n} these functions are *polynomials*

$$h_i = \mathcal{H}_i(p, q).$$

(c) Level sets

If $|\partial^2 S / \partial q \, \partial h| \neq 0$ along M_h, then M_h is an Abelian variety (i.e. a complex torus such that $M_h \cap C^{2n}$ is defined by polynomial equations)*. (Remark: If $|\partial^2 S / \partial q \, \partial h|$ vanishes somewhere along $M|_{h-c_0}$ then this level set is singular and is in fact a "generalized" Abelian variety, but we will not go into that here.)

(d) Irreducible components of $M - C^{2n}$

When d is odd, $M - C^{2n} = A_{(n)}(\tilde{M}|_{\xi_1 - \infty})$ consists of one irreducible component; when d is even $M - C^{2n}$ consists of two irreducible components. (Note that $\tilde{M}|_{\xi_1 - \infty}$ is just the symmetric product of $(n - 1)$ copies of the curve.)

(e) Principal balances

Let $G = \alpha_1 \mathcal{H}_1 + \cdots + \alpha_n \mathcal{H}_n$ be any linear combination of the integrals. This generates a flow which commutes with that of any $\mathcal{H}_i$ on any level set. The principal balance submanifolds for G are contained in $M - C^{2n}$. If $\mathcal{B}$ is a connected component of $M - C^{2n}$, then it contains a principal submanifold for G, $\mathcal{B}_G$, which is open and dense in $\mathcal{B}$. Moreover, if G_1 and G_2 are two integrals, then G_1 and G_2 have orbits which both intersect $\mathcal{B}_{G_1} \cap \mathcal{B}_{G_2}$ and have the same leading exponents at those points of intersection. (It is not meaningful to compare the leading coefficients of the respective series, since the flows are different.)

(f) Lower balances

For each commuting flow and for every integer m in the range from $2n - 1$ to n, there exist balances which depend on just m parameters (excluding the origin of time t_0).

---

*Technically, this only shows that $M_h \cap C^{2n}$ is *birational* to an Abelian variety. However, this is sufficient to conclude that M_h can be embedded in some projective space which is the usual definition. For a discussion of this see [18].

(g) Lowest balances

The free parameters for a lowest balance are just $h_1, \ldots, h_n$. There are three types of lowest balance:

(i) Places where the phase trajectory meets the closure of a principal balance submanifold with maximal order of contact. At such points, all of the variables (q, p) blow up. There are $2(d - 4)$ (resp. $d - 3$) such lowest balances for the ξ-odes (counted with multiplicity) when d is even (resp. odd). (Adler and van Moerbeke also derive such a count in their examples when $n = 2$, see [7]).

(ii) Places where the Abel map cannot be inverted (i.e. critical values of the Abel map), and the q variables are finite while the p variables blow up.

(iii) Critical values of the Abel map where all variables blow up. These will be places where M_h is smooth but $A_{(n)}(\tilde{M}_h|_{\xi_1 = \infty})$ is singular.

Proof. (a) From the discussion in section 3.2 we have the following diagram of coordinate transformations

$$\{(h; (\xi_1, \eta_1), \ldots, (\xi_n, \eta_n))\} \in \qquad \overset{\Phi}{\underset{}{\diagup}} \; \tilde{M} \; \overset{A_{(n)}}{\diagdown}$$

$$(q, p) \in C^{2n} \; \overset{}{\underset{L}{\longrightarrow}} \; \{(h; t \;(\mathrm{mod}\; \Lambda'))\} \in M$$

where $\Lambda' = \pi(\Lambda)$. (The map Φ is only defined for $\{\xi_i \neq \infty\}$.) It follows from Liouville's theorem [11] that the map L is locally one to one. Thus condition (a) of definition (3.1) is satisfied. Since the flows evolve linearly in the t_i they exist for all time and stratify M into disjoint orbits. Hence, conditions (c) and (d) in 3.1 are satisfied. $A_{(n)}$: $\tilde{M} \to M$ is a holomorphic map which is locally one to one away from points where $\xi_i = \xi_j$, $\eta_i = -\eta_j$ for some $i \neq j$. Since the level sets $\tilde{M}|_{h=c}$, $M|_{h=c}$ are compact, it follows that $A_{(n)}$ is generically finite to one. $M - C^{2n} = A_{(n)}(\tilde{M}|_{\xi_1 = \infty})$; hence, $\xi_1 = \infty$ defines an analytic hypersurface in $\tilde{M}$. Since $A_{(n)}$ is a finite map it follows that $M - C^{2n}$ is an analytic hypersurface. We will see in part (b) of this theorem that $M - C^{2n}$ consists of either one or two irreducible components so that condition (b) of definition 2.1 is satisfied. Finally, since $M - C^{2n}$ only contains points which lie on trajectories that originate in C^{2n}, it contains no subsets invariant under the flow. Hence, M is minimal.

When $n = g$, theorem 2 states that $A_{(n)}$ is generically one-to-one and so $\tilde{M}$ is itself an augmentation.

(b) From definition 1b,c we have that the separating variables satisfy

$$\begin{array}{ccc}
\eta_1^2 - \xi_1^d - \displaystyle\sum_{k \in J^-} \alpha_k \xi_1^{d-k} & = \\
\vdots \quad \vdots \quad \vdots & \\
\eta_n^2 - \xi_n^d - \displaystyle\sum_{k \in J^-} \alpha_k \xi_n^{d-k} & =
\end{array}
\begin{bmatrix} \xi_1^{d-j_1} & \cdots & \xi_1^{d-j_n} \\ \vdots & & \vdots \\ \xi_n^{d-j_1} & \cdots & \xi_n^{d-j_n} \end{bmatrix}
\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix},$$

where j_i runs over the index set, J, of the free parameter while $J^- =$ the complement of J in $\{1, \ldots, d\}$. Inverting this linear system presents the h_i as symmetric rational functions of the $\{(\xi_i, \eta_i)\}$.

Recall the diagram used in the proof of (a). By definition 3.1d(i) we know that

$$\Phi: \left(\tilde{M}' - \Phi^{-1}(\Sigma)\right) \to C^{2n} - \Sigma$$

is a finite, unramified covering for some codimension two subset Σ. Since Σ is finite and unramified, its domain $\tilde{M}' - \Phi^{-1}(\Sigma)$, can be partitioned into a disjoint union of mutually homeomorphic subdomains,

each of which map $1:1$ onto $C^{2n} - \Sigma$. Let Ω denote one of these subdomains, also called a *fundamental domain* of this cover. Then Φ maps Ω $1:1$ onto $C^{2n} - \Sigma$. Since Φ is an analytic mapping and the h_i are meromorphic functions on $\tilde{M}, h_i|_\Omega$ $i = 1, \ldots, n$ define, by the $1:1$ correspondence, n meromorphic functions on $C^{2n} - \Sigma$. We claim that if Ω' is a different fundamental domain, then $h_i|_\Omega = h_i|_{\Omega'}$. To see this note that the pullback $\Phi^*(h_i|_\Omega)(\xi, \eta) = {}_{df}h_i|_\Omega(\Phi(\xi, \eta))$ equals h_i on Ω and has the property

$$\Phi^*(h_i|_\Omega)|_\Omega = \Phi^*(h_i|_\Omega)|_{\Omega'}$$

for any fundamental domain. However, by the uniqueness of analytic continuation $\Phi^*(h_i|_\Omega) = h_i$.

It follows that the functions $h_i(\xi, \eta)$ project to well defined functions $\mathcal{H}_i(q, p)$ on $C^{2n} - \Sigma$. Moreover, since the h_i are meromorphic on $\tilde{M}$ and Φ is locally analytic, $\mathcal{H}_i(q, p)$ is meromorphic on $C^{2n} - \Sigma$. Since Σ has codimension 2 in C^{2n}, the $\mathcal{H}_i$ are actually meromorphic on *all* of C^{2n} by the Levi Extension Theorem [19, p. 369].

On the other hand,

$$\frac{\partial \mathcal{H}_i}{\partial q} = \frac{\partial h_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial q} + \frac{\partial h_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial q}$$

and

$$\frac{\partial \mathcal{H}_i}{\partial p} = \frac{\partial h_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial p} + \frac{\partial h_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial p}.$$

Since $q = q(\xi)$ is polynomial it follows that $\partial \xi / \partial p = 0$ and $\partial \xi / \partial q$ is an algebraic function.

$$\frac{\partial \eta}{\partial p} = \frac{\partial}{\partial p} \cdot \frac{\partial S}{\partial \xi} = \frac{\partial}{\partial p} \left(\frac{\partial S}{\partial q} \cdot \frac{\partial q}{\partial \xi} \right) = \frac{\partial}{\partial p} \left(p \cdot \frac{\partial q}{\partial \xi} \right) = \frac{\partial q}{\partial \xi},$$

which is also algebraic. Finally $\partial h_i / \partial \xi$ and $\partial h_i / \partial \eta$ are algebraic since $h_i(\xi, \eta)$ is rational. $\partial \mathcal{H}_i / \partial q, \partial \mathcal{H}_i / \partial p$ are thus both meromorphic and have algebraic growth at infinity. Hence, $\partial \mathcal{H}_i / \partial q$ and $\partial \mathcal{H}_i / \partial p$ are rational functions and so is $\mathcal{H}_i$.

In fact the $\mathcal{H}_i$ are polynomials. By definition 3.1d(i), $\Phi^{-1}(C^{2n} - \Sigma)$ is contained in the subset of $\tilde{M}$ where the ξ_j and η_j coordinates are all finite. By the definition of η, the h_i must also be finite on this set. Thus $\mathcal{H}_i(q, p)$ is in fact analytic off of Σ. By Hartogs' theorem, [19, p. 7] $\mathcal{H}_i$ extends to be analytic on all of C^{2n}. But an analytic rational function on C^{2n} is a polynomial.

(c) We adopt the notation used in the proof of (b). It follows from definition 3.1d(iii) that $(\partial \eta / \partial h_i) d\xi$, $i = 1, \ldots, n$, is a frame of n holomorphic differentials which are everywhere independent along the level set $h = c$ in $\tilde{M} - \Phi^{-1}(\Sigma)$. (Note that $\Phi^{-1}(\Sigma)$ in $\tilde{M}$ may have codimension one even though Σ has codimension two in M.) It follows from definition 3.1d(i) that $\Omega \cap \{h = c\}$ in $\tilde{M}$ is mapped by $A_{(n)}$ $1:1$ onto $(M - \Sigma) \cap \{\mathcal{H} = c\}$ where $\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_n)$ is the vector of polynomial integrals described in (b). Thus we can regard the differentials $(dS / dh_i) d\xi|_\Omega$ as a frame of n independent holomorphic differentials on $\{\mathcal{H} = c\} \cap (M - \Sigma)$. By Hartogs' theorem, since Σ has codimension 2, $(\partial S / \partial h_i) d\xi|_\Omega$ extends to be a holomorphic differential on $\mathcal{H} = c$ in M. Moreover, $(\partial S / \partial h_i) d\xi|_\Omega$, $i = 1, \ldots, n$ are independent on $\{\mathcal{H} = c\} \cap (M - \Sigma)$. Hence, $\det(\nabla \partial S / \partial h_1, \ldots, \nabla \partial S / \partial h_n)$ is non-vanishing in a deleted neighborhood of $\Sigma_0 = \Sigma \cap (\mathcal{H} = c)$. By Hartogs' theorem the inverse of this determinant extends to be holomorphic along Σ_0, so that $(\partial S / \partial h_i) d\xi|_\Omega$ extend to an everywhere independent holomorphic frame along M_c. These

differentials are dual to n independent commuting holomorphic vector fields. Therefore M_c is a complex n-torus C^{2n}/Λ' where Λ' is a rank $2n$ sublattice of the periods Λ of $A_{(g)}$ (3.8). Since $M_c \cap C^{2n}$ is defined by the polynomials $\mathcal{H}_i$, M_c is an Abelian variety.

A consequence of this result is that the period lattice, Λ, splits

$$\Lambda = \Lambda' \oplus \Lambda''$$

where Λ'' has rank $2(g-n)$ and $\Lambda'' \subset$ kernel $\Pi \circ A_{(n)}$.

(d) When d is odd, ∞ is a branch point of $\mathcal{R}$ and so $\tilde{M}_h|_{\xi_1 = \infty} \approx \mathcal{R}_h^{(n-1)}$, the $(n-1)$ fold symmetric product of $\mathcal{R}$ with itself; when d is even there are two points over $\xi_1 = \infty$, $\{\infty, \eta_+\}$, $\{\infty, \eta_-\}$, and

$$\tilde{M}_h|_{\xi_1 = \infty} = \tilde{M}_h|_{\substack{\xi_1 = \infty \\ \eta = \eta_+}} \cup \tilde{M}_h|_{\substack{\xi_1 = \infty \\ \eta = \eta_-}}$$

$$\approx \mathcal{R}_h^{(n-1)} \cup \mathcal{R}_h^{(n-1)}.$$

The images $A_{(n)}(\tilde{M}_h|_{(\infty, \eta_+)})$ and $A_{(n)}(\tilde{M}_h|_{(\infty, \eta_-)})$ differ, in $M_h \approx C^n/\Lambda'$, by the translate $\int_{(\infty, \eta_-)}^{(\infty, \eta_+)} (\partial \eta / \partial h) \, d\xi$ which is nonzero for general h. Hence $A_{(n)}(\tilde{M}|_{\xi_1 = \infty})$ consists of one component when d is odd and two when d is even.

(e) The equations for the $\mathcal{H}_i$ flow are given (see (3.11)) by

$$\frac{\partial \xi_j}{\partial t_i} = \left(\frac{\partial \eta}{\partial h}(\xi) \right)_{ij}^{-1}, \quad j = 1, \dots, n.$$

More generally, the Hamiltonian $G = \alpha_1 \mathcal{H}_1 + \cdots + \alpha_n \mathcal{H}_n$ has vector field $\partial/\partial t = \alpha_1 \partial/\partial t_1 + \cdots + \alpha_n \partial/\partial t_n$ and the corresponding odes are

$$\frac{\partial \xi_j}{\partial t} = \sum_{i=1}^{n} \alpha_i \left(\frac{\partial \eta}{\partial h}(\xi) \right)_{ij}^{-1}, \quad j = 1, \dots, n. \tag{3.15}$$

(The right-hand side of (3.15) is a function of $\{\xi, h\}$.) We will consider the case of odd d, and $n = g$ (see (3.12)), the argument for even d or $n < g$ being similar. Near $\tilde{M} - C^{2n}$ we use the variables $\xi_1 = \lambda^{-2}$; $\xi_2, \dots, \xi_n$ arbitrary. For $\xi_{l \geq 2} \neq \infty$ or any other branch point, eqs. (3.15) take the form (variables with a caret $\hat{}$ are omitted)

$$\frac{\partial \lambda}{\partial t} = 1/2 \, p_\alpha(\xi_2, \dots, \xi_n) \frac{\sqrt{1 + \mathcal{O}(\lambda)}}{\prod_{j=2}^{n}(1 - \xi_j \lambda^2)},$$

$$\frac{\partial \xi_l}{\partial t} = \frac{p_\alpha(\lambda^{-2}, \dots, \hat{\xi}_l, \dots, \xi_n)}{\lambda^{-2}} \frac{\eta(\xi_l)}{(1 - \xi_l \lambda^2) \prod_{j \neq 1, l}(\xi_l - \xi_j)}, \tag{3.16}$$

where p_α is a polynomial of degree $(d - 3)/2$ whose coefficients depend on $\alpha = (\alpha_1, \dots, \alpha_n)$. If $\xi_l = \infty$ or a branch point, (3.16) gets modified by changing to a local parameter. The key point is that $(\partial \eta / \partial h)(\xi)$ is *always* finite in a local parameter and therefore any zeroes of $\eta(\xi_l)$ are balanced by zeros of $\partial \xi_l / \partial t$ when we change coordinates.

For generic initial values of ξ_l ($l \geq 2$), the solutions of (3.16) will exhibit the same leading behaviour in t. This is the principal balance submanifold $\mathcal{B}_G$. Geometrically it is characterized as the submanifold of

points $\tilde{M} - C^{2n}$ at which the flow is transverse. In $\mathscr{B}_{G_1} \cap \mathscr{B}_{G_2}$, (3.16), for both G_1 and G_2, has the same form near $\lambda = 0$. Hence the leading exponents are the same.

(f) A new balance is introduced whenever the flow becomes tangent to $\tilde{M} - C^{2n}$. This just means that

$$\frac{\partial \xi_l}{\partial \lambda}\bigg|_{\lambda=0} = \frac{\partial \xi_l / \partial t}{\partial \lambda / \partial t}\bigg|_{\lambda=0} = \frac{2\bar{p}_\alpha(\xi_2,\ldots,\hat{\xi}_l,\ldots,\xi_n)\eta(\xi_l)}{p_\alpha(\xi_2,\ldots,\xi_n)\Pi_{j\neq 1,l}(\xi_l-\xi_j)} = \infty, \tag{3.17}$$

where $\bar{p}_\alpha(\xi_2,\ldots,\hat{\xi}_l,\ldots,\xi_n) = \lim_{\lambda\to 0}p_\alpha(\lambda^{-2},\ldots,\hat{\xi}_l,\ldots,\xi_n)/\lambda^{-2}$. The degree of tangency of the flow to $\tilde{M} - C^{2n}$ equals the order of zero in the denominator of (3.17) which in turn determines the leading exponents for (p,q). The higher the degree of tangency, the larger these exponents are and the smaller the dimension of the corresponding balance. In particular, if we set $\xi_1 = \infty$ and $\xi_2 = \cdots = \xi_s$ we get a balance of dimension $2n - s + 1$. As s ranges from 3 to n, this produces balances of dimension $2n - 2$ down to $n + 1$.

(g) To get the lowest balances, of dimension n, we require, further, that

$$p_\alpha(\xi_2,\ldots,\xi_n)/\bar{p}_\alpha(\xi_2,\ldots,\hat{\xi}_l,\ldots,\xi_n)$$

have a zero in the ξ_j. If we write p_α as a polynomial in ξ_l:

$$p_\alpha(\xi_2,\ldots,\xi_n) = p_\alpha^{(0)} + p_\alpha^{(1)}\xi_l + \cdots + p_\alpha^{(d-3)/2}\xi_l^{(d-3)/2},$$

where $p_\alpha^{(j)} = p_\alpha^{(j)}(\xi_2,\ldots,\hat{\xi}_l,\ldots,\xi_n)$ is a polynomial, then $\bar{p}_\alpha = p_\alpha^{(d-3)/2}$. Thus the lowest balances correspond to points for which

$$\xi_1 = \infty, \quad \xi_2 = \xi_3 = \cdots = \xi_n = \xi, \quad \text{and}$$

$$\underline{P}(\xi) = \xi^{(d-3)/2} + \frac{p_\alpha^{(d-5)/2}}{p_\alpha^{(d-3)/2}}\xi^{(d-5)/2} + \cdots + \frac{p_\alpha^{(0)}}{p_\alpha^{(d-3)/2}} = 0. \tag{3.18}$$

$\underline{P}(\xi)$ has $(d-3)/2$ roots counted with multiplicity. To determine the total number of lowest balances corresponding to all these roots it only remains to find out what η-values are allowed for these roots. The use of the ξ-odes in (3.17) to study balance structure is valid as long as we can invert (3.10). This breaks down, however, whenever we have $\xi_i = \xi_j$, $\eta_i = -\eta_j$ for some $i \neq j$. Therefore for the balances in part g(i) of this theorem we must have $\eta_2 = \cdots = \eta_n = \pm\eta$. Hence, we have $(d-3)$ such lowest balances determined by (3.18). These balances are indeed lowest, since the remaining constants, $h_1,\ldots,h_n$, fix a level set on which the flows are, generally, ergodic.

It is possible that other balances could arise corresponding to points where $\xi_i = \xi_j$ and $\eta_i = -\eta_j$ for some $i \neq j$. Even though the set of such bad points has codimension 1 in $\tilde{M}$, under $A_{(n)}$ it contracts to something of codimension 2 in M. Hence these points cannot produce a new principal balance submanifold. They can, however, correspond to lower balances. There are two cases. If the ξ_i are all finite, it is easy to see from the variable change, $(q(\xi), p(\xi, \eta))$, that q will stay finite while p blows up. On the other hand if $\xi_1 = \infty$ while $\xi_i = \xi_j$, $\eta_i = -\eta_j$ for distinct i, $j \geq 2$ then both q and p blow up. This can occur for $n \geq 3$. It is a consequence of the Riemann Singularity Theorem [17, p. 56] that the Abel map sends such a point to a singularity of $\Theta = A_{(n)}(\tilde{M}_h|_{\xi_1=\infty})$. An orbit crossing Θ at such a singularity will generally have a different pole structure than an orbit crossing Θ at a smooth point. Fig. 4 depicts this for $n = 3$.
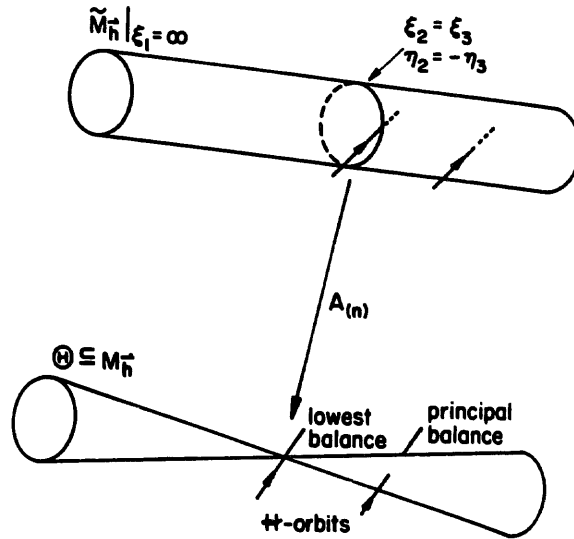
Fig. 4. The origin of lowest balance in theorem 3.3g(iii) for $n = 3$. The map $A_{(n)}$ blows down the fiber $\xi_2 = \xi_3, \eta_2 = -\eta_3$ to a singular point on the hypersurface at infinity, Θ.

3.5. *Examples continued*

To illustrate theorem 3, we extract the various balances for the two examples in section 3.2 for $n = 2$ by using the differential equations on $\tilde{M}_h$ in (3.12) and (3.13).

1. *Gelfand–Dikii*, $\mathcal{H}_1$

There is a single principal balance in which the leading behavior of ξ_1 is given by

$$d\xi_1/dt_1 = 2\xi_1^{3/2}$$

or $\xi_1 = t^{-2}$. The exponents f_i for q_i are therefore both 2, while the p_i diverge with exponents $g_i = (5, 3)$, (cf. (A.3)). Once the ξ-equations are solved, the momenta p_i can always be found from

$$\sum_{i=1}^{n} \frac{\partial q_i}{\partial \xi_j} p_i = \eta_j.$$

At the lowest balance, $\xi_i \sim a_i t^{-2}$, $i = 1,2$, and there are two distinct solutions for a_i that may be characterized by $a_1/a_2 = e^{\pm 2\pi i/3}$. The sign ambiguity in $\eta(\xi)$ never manifests itself. This count is in accord with part g(i) of theorem 3. However, since the q_i are symmetric in the ξ_j a single balance is obtained when we pass to (q, p) variables, viz.,

$$q_1 \sim 3t^{-2}, \quad q_2 \sim 9t^{-4}.$$

The corresponding exponents for p_i are $g_i = (5, 3)$

2. *Gelfand–Dikii*, $\mathcal{H}_2$

The principal balance is now $\xi_1 \sim \xi_0^{-2} t^{-2}, \xi_2 \sim \xi_0 + \mathcal{O}(t)$. Note that the exponents are unchanged from $\mathcal{H}_1$ but the coefficients differ, (see part e of theorem 3).

The lowest balance with q_i diverging follows from the solutions,

$$\xi_1 \sim h_2^{-1} t^{-4}, \qquad \xi_2 \sim \pm 2\sqrt{h_1}\, t.$$

There are two balances in (q, p) since the sign ambiguity in ξ_2 persists but the exponents are identical and equal to $f_i = (4, 3)$ and $g_i = (10, 6)$.

The other lowest balance is obtained only where the Abel map is noninvertible, i.e., $(\xi_i, \eta_i) = (0, \sqrt{h_2}), (\xi_2, \eta_2) = (0, -\sqrt{h_2})$. (When $\eta_1 = \eta_2$ the ξ-equations continue to have a solution around 0 but the p_i are analytic.) One then finds

$$\xi_{1,2} = \pm \sqrt{h_2}\, t + \mathcal{O}(t^2)$$

and therefore

$$q_{1,2} \sim \mathcal{O}(t^2), \qquad p_1 \sim \mathcal{O}(t), \qquad p_2 \sim -t^{-1}.$$

There are again two solutions to the ξ-equations implicit in the square root and two balances in (q, p) variables with the only divergence being in p_2.

If we study the flow $\alpha \mathcal{H}_1 + \mathcal{H}_2$, then, as α varies, the degenerate balance just mentioned runs over the entire fiber on which the Abel map is non-invertible, i.e., $\xi_i = \alpha$, $\eta_i = (\eta(\alpha), -\eta(\alpha))$. The ξ-equations yield $\xi_1 = h_2^{-1} t^{-2}$, $\xi_2 = \alpha \pm \sqrt{h_2}\, t$ and $f_i = (4, 4)$, $g_i = (10, 6)$.

3. Hénon–Heiles, $\mathcal{H}_1$

In the notation of this section, $\mathcal{H}_1$ is twice the Hamiltonian in (2.23) while $\mathcal{H}_2$ is one-half of (2.30). There are two principal balances,

$$\xi_1 = \pm i 2^{-1/2} t^{-1},$$

which are reflected in the (q, p) variables since $q_1 = i\xi_1 \xi_2$.

At the lowest balance, we have $\xi_{1,2} \sim a_{1,2} t^{-1}$ for each of the four possible sign choices for $\eta_{1,2}$. In each case there are two further solutions that are fixed by specifying $a_1^2 / a_2^2 = e^{\pm 2\pi i/3}$. Since the degree of $\eta^2(\xi)$ is 8, these 8 solutions verify the count in part g(i) of theorem 3. They give rise to only two distinct solutions in (q, p) variables (see (2.25))

$$q_i \sim \pm \tfrac{3}{2} i t^{-2}, \qquad q_2 = \tfrac{3}{4} t^{-2}.$$

4. Hénon–Heiles, $\mathcal{H}_2$

The principal balance yields

$$\xi_1 \sim \pm i 2^{-1/2} \xi_0^{-2} t^{-1}, \qquad \xi_2 \sim \xi_0 + \mathcal{O}(t),$$

and the same leading exponents for (q, p) as for $\mathcal{H}_1$, but with different coefficients.

The first lowest balance is

$$\xi_1 \sim \pm 3 i t^{-3} / (4\sqrt{2}\, h_2), \qquad \xi_2 = \pm 2\sqrt{h_2}\, t,$$

where the $(\pm)$ are independent. The exponents for q_i are $(2, 6)$ and for p_i, $(5, 9)$.

The other lower balance has $\xi_{1,2} = a_{1,2}t$ and $\eta_1 = -\eta_2 = \pm \sqrt{h_2}$. There are two solutions for each value of η_1. For (q, p) one obtains

$$q_1 \sim \pm \frac{4i}{3}h_2 t^2, \quad q_2 \sim -\frac{2h_2}{3}t^2,$$

$$p_1 \sim \mp i\tfrac{1}{8}t^{-1} + \mathcal{O}(t), \quad p_2 \sim \tfrac{1}{8}t^{-1} + \mathcal{O}(t).$$

The number of lowest balances in the ξ-equations, appears to fall short of what is expected from theorem 3. This is because the point $\xi_2 = 0$ must be counted with multiplicity two. This becomes obvious when we consider the flow generated by $\mathcal{H}_1 + \alpha\mathcal{H}_2$. The exponents for the lower balance in which ξ_1 diverges actually change to

$$\xi_1 = \pm it^{-2}/\big(2\sqrt{2}\,(\pm\sqrt{\alpha})(\pm\eta_2(\xi_2 = \sqrt{\alpha}))\big), \quad \xi_2 = \pm\sqrt{\alpha} \pm 2\eta_2(\xi_2 = \sqrt{\alpha})t$$

and there are now 8 independent choices of sign. The point $\xi_2 = 0$ has split into $\xi_2 = \pm\sqrt{\alpha}$. The signs on $\sqrt{\alpha}$ and η_2 are all manifest in the expressions for q_i. The additional sign freedom on i which comes from η_1 appears in p_i.

3.6. *Bounds for the degrees of integrals in an h.s. system*

Definition 3. If $h_i = \alpha_{j_i}$ in the separating polynomial (see definition 3.1)

$$\eta^2 = \xi^d + \alpha_1\xi^{d-1} + \cdots + \alpha_d$$

then we assign the *formal degrees*, j_i, to the integral $\mathcal{H}_i(p,q)$.

In this section we will bound the degree of $\mathcal{H}_j$ in terms of this formal degree:

$$\deg_t \mathcal{H}_i(p,q) \le (j_i + 2)(\deg_t \xi),$$

where p, q and $\xi \equiv \xi_1$ are weighted according to the leading exponents in the Laurent series at a principal balance. (Recall from theorem 3e that $\deg_+ \xi$ at the principal balance is the same for *all* flows so we do not need to associate a particular flow with t.)

Recall from the proof of theorem 3b that we derive an expression for $h_i = h_i(\xi, \eta)$ by inverting the system

$$\begin{bmatrix} \eta_1^2 - \xi_1^d - \sum\limits_{k \in J^-} \alpha_k\xi_1^{d-k} \\ \vdots \\ \vdots \\ \eta_n^2 - \xi_n^d - \sum\limits_{k \in J^-} \alpha_k\xi_n^{d-k} \end{bmatrix} = \begin{bmatrix} \xi_1^{d-j_1} & \cdots & \xi_1^{d-j_n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \xi_n^{d-j_1} & \cdots & \xi_n^{d-j_n} \end{bmatrix}\begin{bmatrix} h_1 \\ \vdots \\ \vdots \\ h_n \end{bmatrix}. \tag{3.19}$$

From this, we see that when we take $\xi \to \infty$ with $\xi_i \sim \xi$ and $\eta_i^2 \sim \xi^d$ for *all* i, (which is the correct asymptotic behaviour for h_j to be constant on the level set of an h.s. system) then we have that

$$h_i(\xi, \eta) \sim \xi^{j_i},$$

i.e. h_i grows as a power = formal degree. (Note that we regard ξ, η as independent variables on $\tilde{M}$ even though we relate their growths.)

To bound the polynomial degree of $\mathcal{H}_i(p,q)$ it will suffice to bound the growth of the monomials in $\mathcal{H}_i$ as a function of ξ near the principal balance submanifold. From (3.19) one sees that each h_i has the form

$$h_i(\xi,\eta) = \sum_{j=1}^{n} \alpha_j(\xi)\eta_j^2 + \beta(\xi),$$

$$(3.20)$$

where $\alpha_j(\xi), \beta(\xi)$ are rational functions of ξ. Changing variables from (η, ξ) to (p,q) we have

$$\eta_j = \frac{\partial S}{\partial \xi_j} = \sum_{l=1}^{n} \frac{\partial S}{\partial q_l}\frac{\partial q_l}{\partial \xi_j} = \sum_{l=1}^{n} p_l\frac{\partial q_l}{\partial \xi_j}.$$

$$(3.21)$$

Inserting (3.21) into (3.20) yields

$$\mathcal{H}_i(p,\xi) = \sum_{lk}\left(\sum_j \alpha_j(\xi)\frac{\partial q_l}{\partial \xi_j}\frac{\partial q_k}{\partial \xi_j}\right)p_l p_k + \beta(\xi).$$

$$(3.22)$$

The coefficients, which are symmetric functions of ξ_i, must in fact be polynomial functions of q:

$$\mathcal{H}_i(p,q) = \sum_{l,k} A_{lk}(q)p_l p_k + \beta(q).$$

$$(3.23)$$

The $A_{lk}(q)$ and $\beta(q)$ are polynomial by theorem 3b.

We first observe that along the diagonal, $\xi_j = \xi$ for all j, every monomial in $\mathcal{H}_i(p,q)$, when viewed as a function of ξ, goes like $\leq \xi^{j_i}$ as $\xi \to \infty$. Observe further that in this case, by definition 3.1d(ii), all monomials in (3.21) have a common degree bound, $\xi^{d/2}$, independent of j. Still working along the asymptotic diagonal, $\xi_j = \xi \to \infty$, every monomial in (3.23) has a common degree bound of ξ^{j_i} which it inherits from (3.22) since (3.21) preserves degrees.

If we now desire to weight p,q according to a principal balance, we take $\xi_1 \to \infty$ while $\xi_2,\ldots,\xi_n$ are finite. The monomials in η_1 in (3.21) continue to have a common degree bound of $\xi_1^{d/2}$. For $\eta_{j\geq 2}$ we are differentiating with respect to a ξ_j which is finite, not diverging, hence the common degree bound is one factor of ξ_1 larger than before, or $\xi_1^{d/2+1}$. When we return to the monomials involving $p_l p_k$ in (3.23) we must allow for two more factors of ξ_1 due to the less stringent bound on the monomials in $\eta_{j\geq 2}$. The behavior of A_{lk} is no worse than before and does not upset these estimates, since it is polynomial in q and restricting $\xi_{j\geq 2}$ to be finite can only lessen its degree of divergence not augment it. Therefore we have for the degree of each monomial in (3.23) at a principal balance a bound, $\deg_t \xi_1^{j_i+2}$. Therefore

$$\deg_t \mathcal{H}_i \leq (j_i + 2)(\deg_t \xi_1).$$

$$(3.24)$$

Remark. When we weight (p,q) according to lowest balance of $\mathcal{H}_1$ the bound in (3.24) becomes

$$\deg_t \mathcal{H}_i \leq \deg_t \xi^{j_i}$$

and $\{j_i\}_{i=1}^{n}$ are just the lowest balance resonance degrees of $\mathcal{H}_1$.

Recall from theorem 3 and the ξ-odes that all $\xi_i \to \infty$ with the same degree at the lowest balance for $\mathcal{H}_1$. Hence in contrast to (3.24) all η_i have the same degree and we obtain j_i rather than $j_i + 2$ in our bound.

By expanding $\eta(\xi_i)$ for large ξ_i it is clear that for the lowest balance of the $\mathcal{H}_1$-flow the free constants in the ξ-odes are just the h_i and that they enter with the appropriate degree.

4. Solving h.s. systems

We now exploit the structure of (h.s.) systems detailed in section 3 to design an algorithm that will lead to bounds on the degrees of the integrals in a finite number of operations. (Recall from theorem 3.3 that the integrals must be polynomial.) Along the way we will note how the pole series also reduce the calculation of the transformation $q \to q(\xi)$ to a finite search, and then show by means of an example how additional information can be gleaned from the Hamilton–Jacobi expansion. Our algorithm can, of course, be regarded as a test for h.s., that is, if it fails, the system in question is not h.s. We will utilize the following theorem that follows readily from the results of section 3.

Theorem 1.
a. When d, the degree of η^2, is even, then $j_1 = \overline{\deg}\,\mathcal{H}_1 = (d+4)/2$ and $\deg_t \xi_1 = 1$ whereas if d is odd, $\overline{\deg}\,\mathcal{H}_1 = (d+3)/2$ and $\deg_t \xi_1 = 2$. Here $\overline{\deg}$ is the formal degree of h_i in η^2 defined in section 3.6 and $\deg_t$ is the exponent with which the indicated variable blows up at a principal balance.

b.

$$\overline{\deg}\,\mathcal{H}_1 \ \deg_t \xi = 1 + \deg_t (S)$$

$$= 1 + \sup\,(\deg_t p_i + \deg_t q_i).$$

c. $\overline{\deg}\,\mathcal{H}_j \ \deg_t \xi = \rho_j(\mathcal{H}_1)$ where ρ_j are the resonance degrees $(\rho_i < \rho_j$ if $i < j)$ for the lowest balance of the $\mathcal{H}_1$ flow.

Proof. The Abel map can be inverted perturbatively near the principal balance $(\xi_1 \to \infty, \xi_{i \geq 2}$ finite), for the $\mathcal{H}_1$ flow. The dominant term is found from

$$t_1 \sim \int^{\xi_1} \frac{\xi^{d-j_1}}{\eta}\,d\xi + \text{finite}$$

$$\sim \xi_1^{d/2 - j_1 + 1} \tag{4.1}$$

or

$$\xi_1 \sim t_1^{-1/(j_1 - d/2 - 1)},$$

$(j_1 - d/2 - 1 > 0$ by definition 3.1c). Since we require the series (1.2) to be Laurent and the q_k are symmetric functions of the ξ_j it follows that

$$q = q\left(\ldots \xi_i^{j_1 - d/2 - 1} \ldots\right), \quad d \text{ even,}$$

$$q = q\left(\ldots \xi_i^{2j_1 - d - 2} \ldots\right), \quad d \text{ odd.}$$

(The second line follows because for d odd the exponent of ξ_1 in (4.1) is half integer yet we know q is a polynomial.)

However, in either case the map Φ: $\tilde{M} \to C^{2n}$ (see the proof of theorem 3.3a) given by $(h;(\xi,\eta)) \to (q(\xi), p(\xi,\eta))$ ramifies to order $j_1 - d/2 - 2$ (resp. $2j_1 - d - 3$) along the locus $\xi_1 = 0$. However, by definition 3.1d(i), Φ cannot ramify in codimension 1. Hence $j_1 - d/2 - 1 = 1$, (resp. $2j_1 - d - 2 = 1$) which together with (4.1) establishes (a).

Part (b) follows from the expression for S in definition 3.1 which near a principal balance reads, $S \sim \xi_1^{d/2+1}$. Eliminating d with (4.1) gives part (b), the second line of which follows from the nondegeneracy assumption definition 3.1d(ii).

Finally part (c) just restates the last remark in section 3.6.

Remark. The nonramification condition, definition 3.1d(i), was an essential ingredient in our proof, but may be stronger than what we actually need. Assume that $a = j_1 - d/2 - 1$ in (4.1) is greater than 1, but that q is a function of ξ_i^a or ξ_i^{2a} as is necessary to eliminate fractional powers of t. If the substitution $\mu = \xi^a$, left the form of (3.5b) invariant i.e., $t_i \sim \sum_j \int^{\mu_i} \mu^{n_i}/\lambda \, d\mu$ where λ^2 is polynomial in μ and $n_i \in Z$, then it would mean that the ξ_i were inappropriate variables and the μ_i should be used instead. If conversely, there were fractional powers of μ in λ^2 then as the iterative inversion in (4.1) is continued beyond the leading term fractional powers of t will enter. Unless there are "accidental" cancellations, fractional powers will enter $q(t)$ even though the leading term is Laurent, which is a contradiction. Clearly the last step is not rigorous but is indicative of what we believe the essential mechanism to be.

Theorem 1 maps out the steps to follow in deciding if some $\mathcal{H}$ is h.s. Whether d is odd or even can be ascertained from whether there are one or two principal balances differing by $\pm$, (cf. remark 4 to definition 3.1). Therefore one knows whether $\xi \sim t^{-1}$ or t^{-2} since all flows have the same principal balances. The degrees of the polynomials in the separating transformation $q \to q(\xi)$ are then known and something of their form can be guessed from where the first free constants fall in the q pole series. The degree d of η^2 is also known by computing the dominant term in $S(q)$. Clearly with a finite amount of calculation $q(\xi)$ can be found.

The second alternative is to use part (b) and determine the degree of $\mathcal{H}_1$. Since it too must be polynomial, a finite calculation will yield an explicit answer. Its lowest balance series then generate the remaining degrees.

One can clearly imagine cases in which the "finite" enumerations mentioned above can be quite tedious. Clearly more information about $q(\xi)$ and $\eta^2(\xi)$ is contained in $S(q)$ which can be computed systematically around a principal balance by expanding the Hamilton–Jacobi equation. We have not been able to organize this information cleanly and state a theorem so we conclude with an example which illustrates some of the subtleties involved.

Near a principal balance the action for any h.s. system becomes, (neglecting numerical constants),

$$S = \xi_1^{d/2+1} + \cdots h_1\xi_1^{d/2-j_1+1} + \cdots + \sum_{j \geq 2} \int^{\xi_j} \eta. \tag{4.2}$$

The first "free" function enters the expansion at order t^0 (cf. (2.7)) followed by h_1 at order t^1 and h_i at order $\xi_1^{j_1-j_i}$. After changing back to the q-variables in which the expansion (of (2.6)) is done, clearly the diverging terms reflect only the fixed terms in η^2 plus the transformation $q(\xi)$. Terms that vanish with t can come from any of the ξ_i.

It is informative, now, to look back at an earlier example in section 2.3.4, the $\mathcal{H}_1$ flow for Hénon–Heiles, to see how it corresponds to (2.2). In particular the free function lies in the kernel of (2.5) linearized about the diverging terms. It is a function of $q_1/(-q_2)^{1/2}$ which is precisely ξ_2 when $t = 0$. The

free constant h_1 enters the expansion since it appears explicitly as the energy parameter in the Hamilton–Jacobi equation. The next integral h_2 enters to higher order through derivatives of f.

From these remarks, it is not at all obvious how h_1 enters the Hamilton–Jacobi expansion for $\mathscr{H}_2$ since it is h_2 that occurs explicitly and h_1 only enters through $f^{\xi_2}\eta$. We therefore consider as an example the $\mathscr{H}_2$ flow for Gelfand–Dikii when $n = 2$, (3.2b).

The ξ-ode, (3.11), expanded around a principal balance, shows that $(\xi_0 \equiv \xi_2^0, \eta_0 \equiv \eta_2^0)$

$$\xi_1 = (\xi_0 t)^{-2}(1 - 2(\eta_0/\xi_0)t + \cdots),$$

$$\xi_2 = \xi_0(1 + 2(\eta_0/\xi_0)t + \cdots),$$

(4.3)

and therefore the Laurent series for (3.2b) begin as

$$q_1 = t^{-2}(\xi_0^{-2} + 2\xi_0^{-3}\eta_0 t + \cdots),$$

$$q_2 = t^{-2}(\xi_0^{-1} + \mathcal{O}(t^2) + \cdots),$$

$$p_1 = t^{-5}(\xi_0^{-5} - 5\xi_0^{-6}\eta_0 t + \cdots),$$

$$p_2 = t^{-3}(-\xi_0^{-3} + 3\xi_0^{-4}\eta_0 t + \cdots).$$

(4.4)

As described in the appendix, the leading terms in the Hamilton–Jacobi expansion of S in (2.6) can be determined by solving (4.4) for t, ξ_0 in terms of q_i and then finding p as a function of q, viz.,

$$\xi_0 \sim q_2 q_1^{-1}, \quad t \sim q_1^{1/2} q_2^{-1}$$

and

$$p_1 = \frac{\partial S}{\partial q_1} \sim q_1^{5/2}, \quad p_2 = \frac{\partial S}{\partial q_2} \sim -q_1^{3/2}.$$

Hence,

$$S = \tfrac{2}{7}q_1^{7/2} - q_2 q_1^{3/2} + \cdots.$$

(4.5)

Therefore $\deg_t S = 7, \deg_t \xi = 2$ from (4.3) and $\overline{\deg}\,\mathscr{H}_1 = 4$ from (3.1a), which taken together verify parts a, b of theorem 1. The lowest balance resonances for $\mathscr{H}_1$ are 8 and 10 which agree with the degrees of $\mathscr{H}_{1,2}$ in (3.2a, b) if the (q, p) are weighted according to (4.4).

If the iteration of S is continued from (4.5) to order t^1 one obtains, setting $x = q_1^{-1/2}$, $y = q_2/q_1$,

$$S = \tfrac{2}{7}x^{-7} - yx^{-5} + \tfrac{1}{4}y^2 x^{-3} - \tfrac{1}{8}y^3 x^{-1} + f(y) + \left(\tfrac{29}{64}y^4 + \frac{h_2 - (\partial f/\partial y)^2}{y}\right)x + \cdots.$$

(4.6)

The free function f is the kernel of the linearized Hamilton–Jacobi equation and as expected from (4.2) $y = \xi_0 + \mathcal{O}(t)$. On general grounds, we expect S to be finite when $y \to 0$ and therefore the $\mathcal{O}(t)$ term implies,

$$(\partial f/\partial y)^2 = h_2 + h_1 y + \mathcal{O}(y^2).$$

(4.7)

At this stage, of course, we can only say that (4.6) allows us to introduce a new constant h_1 but does not require it. If the principal balance pole series are substituted into $p_i = \partial S/\partial q_i$, one would find indeed that $h_1 \neq 0$.

Several other remarks about (4.6) are in order. Since we know that (3.2b) is the second Hamiltonian in KdV, h_2 cannot appear in (4.6) till $\mathcal{O}(t^3)$; so, it was necessary that the explicit h_2 that appeared in the Hamilton–Jacobi equation and which entered (4.6) at $\mathcal{O}(t)$, cancel. We do not expect higher order terms in (4.6) to completely fix f, however it does appear that the Hamilton–Jacobi equation for $\mathcal{H}_2$ carries more information about f than the analogous equation for $\mathcal{H}_1$. There is certain symmetry here, since, if one had $\mathcal{H}_1$ one could proceed directly to part c of theorem 1.

5. Conclusion

In the three previous sections we have shown how to use the flows plus the Hamiltonian structure to complete the phase space, examined the consequences of hyperelliptic separability, and by matching series and Hamilton–Jacobi information to this case, derived bounds on the polynomial Hamiltonians in involution. Conspicuously lacking has been a general statement about the relation of the Painlevé property to integrability. Since we have nothing more than a general idea as to a mechanism by which integrability arises, we have reserved these remarks for the conclusion.

Recall the one-variable Riccati example of section 1.2. The augmented manifold was just the Riemann sphere, and therefore compact. General arguments using compactness and the analyticity of x_t on M then suffice to establish the functional form of $x_t(x_0)$ i.e., that it be fractional linear. Similar conclusions may be drawn if one replaces compactness of M by a polynomial growth condition on x_t.

This can be applied to the 1-parameter family of solutions (2.2) to conclude that $x(t, x_0)$ is a rational function of x_0, with coefficients depending on t, if x is polynomially bounded for large x_0. This is analogous to the addition theorems for abelian functions [19].

Thus either M can be compactified as just outlined, or $x(t, x_0)$ is a transcendental function of x_0. In the latter case we would like to claim that there is an entire function $f(t, x_t) = f(0, x_0)$. An example will illustrate how this condition can come about. Consider

$$\mathcal{H} = \tfrac{1}{2}p^2 q^2.$$

Then

$$
\begin{aligned}
p_t &= p_0 \, e^{-p_0 q_0 t}, \\
q_t &= q_0 \, e^{p_0 q_0 t}
\end{aligned}
\tag{5.1}
$$

and $p_t q_t = p_0 q_0$. Observe that although the finite time map is entire with order 2 (i.e., the maximum modulus grows as $\exp(\mathrm{cst}\, r^2)$ for $|q_0|^2 + |p_0|^2 < r^2$) this order does not grow with t. It is no accident that p_t, q_t are transcendental in a quantity which is constant in time and hence compositions in time preserve finite order growth in the phase variables [21]. Integrals are forced to exist, we believe, for any one parameter group of entire non-zero order maps.

These notions generalize to maps meromorphic in time as is appropriate to the Painlevé context since arguments about growth can be rephrased in terms of distributions of zeros. Hence the elliptic functions and other algebraically separable systems also illustrate our point. They define time maps which are rational only on an energy surface but essential as functions of energy.

A close examination of continuous time maps from C^N to C^N seems a logical first step in establishing whether the mechanism we have outlined forces integrals to exist. Entire maps are also tangentially related to the following refinement of the Painlevé test:

Conjecture. If a polynomial system of n differential equations has the Painlevé property and has only principal balances then it is ode integrable, i.e. x_t lies on the intersection of $n-1$ level surfaces defined by entire functions. The converse is clearly trivial since any solution carries with it the maximal number of free constants.

If one grants that the Painlevé test does work, one would next inquire as to how the actual integration could be facilitated by the Painlevé analysis. The explicit construction of the augmented manifold only solves the initial value problem, in an abstract sense, if the manifold is compact. Otherwise, it relates the solutions of pairs of polynomial Painlevé Hamiltonians via the transformation to variables in the principal patch(es) at infinity. Our transition functions resemble Bäcklund transformations which are rational functions relating pairs of solutions to the same integrable system [9].

Knowledge of the augmented manifold can greatly facilitate numerical integration of the system in a natural way. Clearly, in any numerical scheme when the dependent variables get large, one wants a $1:1$ transformation to new coordinates which are small. Systematizing the requirements for the various variable changes leads one to define essentially M as we did in (2.1).

One natural extension of our results would be to consider the more general notion of *algebraic separability* in which the restriction of hyperellipticity is dropped and η in definition 1b is taken to be any of the k branches of

$$\eta^k + \alpha_1(\xi)\eta^{k-1} + \cdots + \alpha_k(\xi) = 0,$$

where the $\alpha_i(\xi)$ are polynomials in ξ and the other conditions in definition 1 are appropriately modified. With the proper formulation most of our results will extend to algebraically separable systems.

In another direction, one could consider Hamiltonian systems on more general symplectic, or even Poisson, manifolds, such as the Neumann system or the Toda lattice. Much significant work on this has already been carried forth by Adler, Haine and van Moerbeke [6, 7, 10]. For two degree of freedom systems they use Laurent time series together with both polynomial integrals to embed a level set into a large ambient space where, by methods of algebraic geometry, they show it completes naturally to an abelian surface. In this setting the Hamilton–Jacobi equation (on a manifold) should still provide an augmentation of the symplectic leaves. The structure of balances for these more general systems should provide information helpful in determining the existence of integrals.

It is well known that the Painlevé property is not necessary for integrability since any one degree of freedom Hamiltonian system with $\mathcal{H} = p^2 + q^m$, $m > 4$, blows up with rational exponents. This was thought to be a particularity of one dimension where any canonical Hamiltonian system can be reduced to quadratures. However several $n = 2$ examples have been found with two polynomial Hamiltonians in involution and fractional exponents [5]. We now explain the pathology in the higher dimensional examples and, in fact, show that one of them is just a degeneration of systems belonging to the Hénon–Heiles hierarchy introduced in eq. (3.3a, b).

In section 2.3.4(d) we remarked for $n = 2$ Hénon–Heiles that there were formal balances for the $\mathcal{H}_2$ flow with $q_i \sim t_2^{-2/3}$, $p_i \sim t_2^{-1}$ but with no solution for the coefficients. These fractional exponents become

real balances if we examine the $n = 1$ system in (ξ, η) variables

$$\mathcal{H}_2 = \eta^2 + \tfrac{1}{2}\xi^8 - h_1\xi^2, \tag{5.2}$$

where h_1 is just a parameter. It is trivial to see that ξ diverges as $\sim t_2^{-1/3}$ which is consistent with the formal balance, $q_i \sim t_2^{-2/3}$ by (3.3b). The flow generated by (5.2) as an $n = 1$ system is identical to the projection of the full system onto $\xi_1 = $ infinity. This may be seen by taking the appropriate limit in (3.13a) but is obvious geometrically since $\tilde{M}$ in theorem 3.3 is the direct product of 2 copies of the separating curve. Whenever the genus exceeds n, one should expect to see fractional exponents unless there is a special symmetry, (c.f. theorem 3.2).

The example in [5] of interest is

$$\mathcal{H}_2 \equiv \tfrac{1}{2}\left(p_1^2 + p_2^2 \right) + q_2^5 + q_2^3 q_1^2 + \tfrac{3}{16} q_2 q_1^4. \tag{5.3}$$

It separates under

$$q_1 = i\xi_1\xi_2, \qquad q_2 = -\tfrac{1}{2}\left(\xi_1^2 + \xi_2^2 \right)$$

into two copies of

$$\tfrac{1}{2}\eta^2 = -\tfrac{1}{32}\xi^{12} + h_2\xi^2 + h_3, \tag{5.4}$$

where h_2 is the value of $\mathcal{H}_2$ and h_3 represents the other integral in involution. Eq. (5.4) after trivial rescalings is identical to $n = 3$ Hénon–Heiles with $h_1 = 0$, (3.3a). Since the $\mathcal{H}_1$ flow is omitted, expansion of (3.5a) shows

$$\xi \sim t_2^{-1/3},$$

which correlates with the exponents $q_i \sim t_2^{-2/3}$ found in [5].

Our geometric interpretation of this example follows what was said above, namely that it is the projection of the $n = 3$ Hénon–Heiles flow onto the hypersurface $\xi_3 = $ infinity with the parameter $h_1 = 0$. (In fact (5.3) should be part of a one parameter family of Hamiltonians.) To be more explicit, write (5.4) for ξ_1, ξ_2, solve for $\mathcal{H}_2, \mathcal{H}_3$ and write the t_2, t_3 flow equations for ξ_1, ξ_2. These are identical to the $\xi_3 = $ infinity limit of (3.13) for $n = 3$.

Appendix

Formal variables change to the principal balance constants

In this appendix we demonstrate how to partition the pole constants at a principal balance into conjugate pairs involving a "phase like" and "action like" constant. The sum of the corresponding resonance exponents then assumes a simple form. Strictly speaking for what follows we will need an invertibility assumption namely, that the Jacobian from $\{q, p\}$ to the pole constants $t_0, c_1, \ldots, c_{2n-1}$ is non zero around $c_i = 0$. We will show immediately in lemma A.1 that this Jacobian may be computed exactly once the series are known to an order which includes all the constants. Since we believe that the finite time map is well defined through infinity and in its neighborhood, our assumption is completely reasonable. To establish this however, requires showing the pole series converge; which in turn requires the Hamilton–Jacobi expansion and transition function to infinity; whose existence follows from the pairing we establish here. Hence to avoid circularity either the invertibility assumption is needed, or a simple check has to be made for each example. We know of no examples in which the Jacobian is not a numerical constant for all $\{c\}$.

Lemma A.1. If the pole series are used to define a formal variable change from $\{q, p\}$ to $t_0, \{c\}$ then the 2-form

$$\omega^{(2)} = \sum_{i=1}^{n} dp_i \wedge dq_i = dt_0 \wedge dE(\{c\}) + \sum_{i,j=1}^{2n-1} \Gamma_{i,j}(\{c\}) dc_i \wedge dc_j, \tag{A.1a}$$

where both E and Γ are polynomial in c_i and are computable from a finite number of terms in the series. (We use E to denote the value of the Hamiltonian $\mathcal{H}(p, q)$.)

Proof. Eq. (A.1) is independent of time since $\omega^{(2)}$ is left invariant by Hamiltonian flows. Since there is a largest negative exponent R in the Laurent series, we need only go up to $\mathcal{O}(t - t_0)^{R+1}$ to be sure of getting all time independent terms in $\omega^{(2)}$. Since each Taylor coefficient in p, q is polynomial in $\{c\}$, Γ must be also. Because $\mathcal{H}$ is autonomous, t_0 occurs only in the combination $(t - t_0)$ and therefore does not enter Γ. Clearly E is a polynomial in c_i and we may compute using Hamilton's equations

$$\sum_{j=1}^{2n-1} \sum_{i=1}^{2n-1} \left(\frac{dp_i}{dt} \frac{\partial q_i}{\partial c_j} - \frac{\partial q_i}{\partial t} \frac{\partial p_i}{\partial c_j} \right) (-) dt_0 \wedge dc_j = \sum_{j=1}^{2n-1} \frac{\partial E}{\partial c_j} dt_0 \wedge dc_j,$$

which yields the first term on the right of (A.1a).

To characterize the pole constants more precisely we will show that it is possible to break them into a "phase like" group, $(t_0, b_1, \ldots, b_{n-1})$ and an "action like" group $(E, a_1, \ldots, a_{n-1})$. Observe that one constant in each pair on the right-hand side of (A.1a) must come from q and the other from p. If the q-constant comes with a nonpositive power of $(t - t_0)$ then the p-constant enters (A.1a) with a nonnegative power of $(t - t_0)$ since the product is indep · of time. To be more precise rewrite

$$\omega^{(2)} = \sum_{1}^{n} dp_{i<} \wedge dq_{i>} + dp_{i>} \wedge dq_{i<} + dp_{i0} \wedge dq_{i0},$$

where the symbols $<$, $>$, 0 stand respectively for negative, positive and 0 powers of t. Expand the

Jacobian determinant from $\{q, p\}$ to the constants in terms of all pairs of $n \times n$ minors. A given q_i or p_i can occur with one and only one of the subsymbols $>$, $<$, 0 in a given product of minors. For each i if we permit ourselves to interchange what we mean by p_i and q_i then there will be at least one nonzero term of the form

$$P = \det\left(\frac{\partial q_{i<,0}}{\partial(t_0, b_j)}\right)\bigg|_{\{b\}=0} \det\left(\frac{\partial p_{i>,0}}{\partial(E, a_j)}\right)\bigg|_{E,\{a\}=0}.$$

Any nonvanishing P serves to distinguish the phase-like from the action like constants so the partitioning may not be unique at this stage though it is in all (h.s.) systems and all other examples we are aware of. We expect that after certain relabelings it is possible to rewrite the second summation in (A.1a) as

$$\sum_1^{2n-1} \Gamma_{ij}(\{c\})\, dc_i \wedge dc_j = \sum_1^{n-1} db_i \wedge da_i + \sum_1^{n-1} \tilde{\Gamma}_{ij}\{(c)\}\, db_i \wedge da_j, \tag{A.1b}$$

where $\tilde{\Gamma}_{ii}(0) = 0$. (In all examples we are aware of $\tilde{\Gamma}_{ij} \equiv 0$.) Recall that $\omega^{(2)}$ is nondegenerate and the variable change from $\{q, p\}$ to the constants is invertible around $\{c\} = 0$. Eq. (A.1b) then defines the pairing between $\{a\}, \{b\}$.

We now have sufficient information to show that at a principal balance it is possible to expand the Hamilton–Jacobi equation as asserted in (2.B). For $p, q \in C^{2n}$ there is a single valued function of $\{q, p\}$ or $t_0, \{a\}, E, \{b\}$ such that,

$$dS = \sum_1^n p_i\, dq_i - E\, dt_0 - \sum_1^{n-1} a_j\, db_j.$$

We seek instead $S(q, a, E)$. By what has been said we can formally invert the $\{q\}$ series to yield $t - t_0, \{b\}$ as functions of $\{q\}$ and then substitute into $p_i(t - t_0, E, \{a\}, \{b\}) = \partial S(q, E, \{a\})/\partial q_i$, and integrate to find S.

Of course it is generally impossible to truncate the pole series consistently so that the variable change is precisely symplectic and one therefore has to work through the Hamilton–Jacobi equation to achieve a consistent truncation. We can expect, however, that the requisite number of free parameters will enter and that when S is truncated, these parameters as functions of q, p will approximate $E, \{a\}$.

The pairing between constants in (A.1b) also gives rise to an inequality on the sum of the resonance degrees of a conjugate (a, b) pair since the free constants must enter the series at or before the place in which they contribute to the variable change in (A.1a,b). Partial eigenvectors associated with resonances are defined as (replacing $t - t_0 \rightarrow t$)

$$\alpha_j = \frac{\partial p}{\partial a_j}\bigg|_{\{a,b\}=0} = t^{-f+\rho_j^a} p_j^0,$$

$$\beta_j = \frac{\partial q}{\partial b_j}\bigg|_{\{a,b\}=0} = t^{-g+\rho_j^b} q_j^0,$$

where we have defined the resonance ρ^b, (resp. ρ^a), to be the first place in which the respective constant enters the q, (resp. p), series.

Since the canonical 2-form is diagonal in $\{q, p\}$ in order for a given conjugate pair a, b, to contribute to (A.1b) there must be a q_k, p_k such that

$$-f_k - g_k + \rho_j^a + \rho_j^b \leq 0. \tag{A.2}$$

An equality is obtained in (A.2) whenever $\omega^{(2)}$ evaluated on the pair of vectors α_j, β_j is nonzero with the q, (resp. p), component of α_j, (resp. p, β_j) zero. However we emphasize that the free constants can enter the series before they are "needed" to satisfy (A.1b). Since we are defining the resonances as the first place a constant enters the appropriate series, the inequality remains a real possibility in (A.2).

The $\mathcal{H}_1$ flow for $n = 2$ KdV is an informative illustration of why some care is required in defining the resonance degrees from the series. It also illustrates certain subtleties in the formal iteration procedure that yields the pole series at a principal balance. The series for $\mathcal{H}_1$ in (3.4a) read [22]

$$q_1 = t^{-2} + \tfrac{1}{2}\alpha - \tfrac{3}{4}\alpha^2 t^2 + \tfrac{1}{2}\beta t^3 - \tfrac{5}{4}\alpha^3 t^4 + \tfrac{3}{8}\alpha\beta t^5 + \tfrac{1}{2}\gamma t^6,$$

$$q_2 = \tfrac{3}{2}\alpha t^{-2} - \tfrac{3}{2}\alpha^2 + \tfrac{3}{4}\beta t - \tfrac{9}{8}\alpha^3 t^2 - \left(\tfrac{9}{4}\gamma + \tfrac{33}{32}\alpha^4\right) t^4,$$

$$p_1 = -t^{-5} + \alpha t^{-3} - \tfrac{1}{8}\beta - \tfrac{1}{2}\alpha^3 t + \tfrac{15}{16}\alpha\beta t^2 + \left(\tfrac{11}{2}\gamma + \tfrac{11}{8}\alpha^4\right) t^3, \tag{A.3}$$

$$p_2 = t^{-3} + \tfrac{3}{4}\alpha^2 t - \tfrac{3}{4}\beta t^2 + \tfrac{5}{2}\alpha^3 t^3 - \tfrac{15}{16}\alpha\beta t^4 - \tfrac{3}{2}\gamma t^5.$$

We therefore find,

$$\omega^{(2)} = dt_0 \wedge dE - \tfrac{27}{16} d\alpha \wedge d\beta,$$

$$E = -\left(\tfrac{81}{4}\gamma + \tfrac{447}{32}\alpha^4\right).$$

By our definition, α is a "phase like constant" and γ and β are action like. Therefore we have $\rho_\alpha = 0, \rho_\beta = 5, \rho_\gamma = 8$, and

$$\rho_\alpha + \rho_\beta = f_2 + g_2,$$

$$\rho_{t_0} + \rho_\gamma = f_1 + g_1.$$

If one defines ρ as the degree of the first correction term which enters any series as is sometimes done in the literature, then the q_1 series sets the degrees of all the constants i.e., $\rho_\beta = 3, \rho_\gamma = 6$ and (A.2) becomes an inequality [22].

Actually this subtlety is associated with a difficulty in naively generating the pole series from the differential equations. If Hamilton's equation are written out for $\mathcal{H}_1$ and linearized about the leading terms, one finds that the Kowalevska determinant for ρ is only 3×3. This is because the q_2 equation reads $\dot{q}_2 = -2q_1 p_2 - 2p_1$ and the derivative term is lower order than the remaining two (t^{-3} vs. t^{-5}), which balance. If one just blindly proceeds using the linearized righthand side of $\dot{q}_2$ as a constraint on the eigenvector, then one has four equations in three unknowns (δq_2 never appears in the other equations) and no solutions.

Clearly a good series does exist and by examining (A.3) one can see where the prescription of retaining only the dominant monomials and linearizing goes wrong. Alternatively one could assign a fictitious leading term of $0t^{-4}$ to q_2. Then the Kowalevska matrix is 4×4, all monomials in Hamilton's equations are the same order and the eigenvectors are obtained by differentiating (A.3) with respect to α, β, and γ at $\alpha = \beta = \gamma = 0$. The roots are $\rho = 2, 5, 8$ and satisfy (A.2).

The $\mathcal{H}_2$ flow for KdV illustrates why an inequality must be allowed in (A.2). The series are given in section 4.1. The phase-like constant of course occurs in the leading coefficients of q with $\rho = 0$ since the system is separable, but an action like constant enters the p-series with a negative power of t. Therefore $-g_k + \rho_1^a < 0$, and $\rho_1^b = 0$.

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