PAINLEVÉ PROPERTY AND INTEGRABILITY

Nicholas ERCOLANI
Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

and

Eric D. SIGGIA
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, NY 14853, USA

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For an n degree of freedom hyperelliptic separable hamiltonian, the pole series with $n+1$ free constants, through the Hamilton–Jacobi equation, bounds the degrees of the n-polynomials in involution. When all the pole series have no fewer than $2n$ constants, the phase space is conjectured to be just the direct product of $2n$ complex lines cut out by $(2n-1)$ integrals.

Exactly solvable or integrable nonlinear systems are exeptional; yet their study has revealed unexpected connections between geometry, analysis, and statistical mechanics [1]. Integrable systems were typically discovered by chance or through techniques specially tailored to the particular problem. There is currently no way of determining whether the most comprehensive approach to nonlinear integrable equations, inverse scattering, will apply short of actually implementing it. Thus a simple objective analytic test is needed for integrability.

Such a procedure was first used, without rigorous justification by Kowalevska [2] who observed that all the known integrable systems, when continued to complex times, were analytic except for isolated poles. Painlevé enumerated all the differential equations of second order whose moveable singularities (i.e., whose location depends on initial data) were only poles [3]. More recently, it was observed that systems solvable by inverse scattering possessed Painlevé's property [4]. Yet, in spite of the many other integrable systems that were shown to be Painlevé [5,6], there is no firm argument as to why the test works or an indication of how to exploit the singularity analysis to yield the integrals. Stated differently, why does the absence of essential singularities and branch points in complex time guarantee that the local patches of level set, which exist for any system of differential equations, combine to become globally defined integrals?

In this article we state a theorem which demonstrates that the singularity analysis provides bounds on the degrees of polynomial integrals for a large class of separable systems. We recast the local singularity data in global geometric terms which provides an intuitive reason for why integrability follows from Painlevé. We formulate a conjecture which permits Liouville integrability [7] to be distinguished from a stronger form ($m-1$ integrals for m equations), by using only local information.

We restrict attention to discrete, autonomous, hamiltonian systems with the hamiltonian H polynomial in the $2n$ conjugate variables $\{q_i, p_i\}$. Virtually all the known examples assume this form once it is realized that any polynomial dependence on time may be incorporated by adding a new degree of freedom. The differential equations define $\{q_i, p_i\}$ for complex times. To implement the Painlevé test, one constructs a formal Laurent series solution

$$q_i \sim t^{-f_i}[1+\dots \mathrm{O}(ct^{\rho})] ,$$
$$p_i \sim t^{-g_i}[1+\dots \mathrm{O}(ct^{\rho})] . \quad (1)$$

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The $\{f_i, g_i\}$ are the leading exponents, and $O(ct^\rho)$ represents up to $(2n-1)$ free constants that enter the series at generally distinct resonance orders $\rho \geqslant 0$. We define a principal balance to be a series (1) with the maximum number, $2n-1$, of free constants (we suppress an additional constant t_0, the origin of time). The lower balances are those with fewer constants which we order by the number present. For a Painlevé system, f_i, g_i, and ρ_j are all integers and there must be at least one principal balance.

Rewrite the canonical two-form $\omega^2 = \Sigma \mathrm{d}p_i \wedge \mathrm{d}q_i$ in terms of the constants at a principal balance by substituting (1). Only a finite number of terms in the series are involved since ω^2 is time independent. The constants may be redefined and partitioned into two groups $\{t_0, c_j\}$, $\{h, \tilde{c}_j\}$, where $j=1,2,\ldots, n-1$ and h is the energy, such that the former (resp. latter) first enters the q (resp. p) series with a non-positive (resp. non-negative) power of t and in addition,

$$\omega^2 = \mathrm{d}t_0 \wedge \mathrm{d}h + \sum_1^{n-1} \mathrm{d}c_j \wedge \mathrm{d}\tilde{c}_j + \sum_1^{2n-2} \Gamma_{lk}(\bar{c})\mathrm{d}\bar{c}_l \wedge \mathrm{d}\bar{c}_k \,, \quad (2)$$

where $\Gamma_{lk}(0)=0$ and $\bar{c}=\{c, \tilde{c}\}$. Note that this establishes a conjugate pairing among the constants. (One may also need to interchange q_j and p_j.) Define a corresponding value of ρ_i (resp. $\tilde{\rho}_i$) by the first correction term in the q (resp. p) series where the associated constant appears. Then

$$\rho_i + \tilde{\rho}_i = f_l + g_l \quad (3)$$

for each i and some $l(i)$ [‡1]. In what follows we need the technical assumption $\Gamma_{lk} \equiv 0$ which is true in all examples we are aware of.

We first state (and prove elsewhere) our theorem on separable systems and then turn to a more general construction which leads to a conjecture as to the mechanism by which the Painlevé property forces integrability, suitably defined.

Definition. For an n degree of freedom hyperelliptic separable h.s. system, the Hamilton–Jacobi equation separates in new canonical variables (ξ_i, η_i) with the properties:

(a) $\{q_i\}$ is a symmetric polynomial function of $\{\xi_i\}$,

(b) η_i^2 is a polynomial in ξ_i of degree $L \geqslant 2n+1$,

(c) the values h_i of the n polynomial integrals $H_i(q, p)$ occur as the coefficients of ξ in η^2 of degree $< L/2$.

Remark. Since the method of separation provides n integrals in involution, a h.s. system may be shown to satisfy the Arnold–Liouville theorem [‡2]. The level set in $\{q_i, p_i\}$ defined by fixing $\{H_i\}$ is topologically a complex n-torus and the flows are quasiperiodic on it. The "angle" coordinates on the torus are defined by the Abel map $t_i = \partial S/\partial h_i$ where the action $S = \Sigma_i \int \eta_i \mathrm{d}\xi_i$ [10]. On a level set of the polynomial $\eta^2(\xi)$ defines a hyperelliptic curve γ. The jacobian of the Abel map is nonzero when $(\xi_i, \eta_i) \neq (\xi_j, \eta_j)$ for all $i \neq j$. Jacobi showed that the Abel map is 1 : 1 away from this locus [11]. Thus the level set is biholomorphic to the nth symmetric product of γ with itself, $\gamma^{(n)}$, off of a locus of codimension two on the level set.

Examples.

(a) For the integrable "Hénon–Heiles" model [5]

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + q_1^2 q_2 + 2q_2^3 \quad (4)$$

separates with

$$q_1 = \mathrm{i}\xi_1\xi_2 \,, \quad q_2 = \tfrac{1}{2}(\xi_1^2 + \xi_2^2) \,,$$

$$\eta_i^2 = -\tfrac{1}{2}\xi_i^8 + 2h\xi_i^2 + g/2 \,, \quad (5)$$

where h and g represent the values of the integrals H and

$$G = 4p_1p_2q_1 - 4p_1^2q_2 + 4q_1^2q_2^2 + q_1^4 \,.$$

(b) For general n the higher stationary solutions in the Korteweg–de Vries hierarchy are separated by replacing $\{q_i\}$ by the elementary symmetric polynomials in $\{\xi_i\}$ and taking [12]

$$\eta_i^2 = \xi_i^{2n+1} + \sum_1^n h_i \xi^{n-i} \,. \quad (6)$$

(For $n=2$, $q_1 = \xi_1 + \xi_2$, $q_2 = \xi_1\xi_2$, $H_1 = -q_1p_2^2 - 2p_2p_1 + 3q_2q_1^2 - q_1^4 - q_2^2$, and $H_2 = 2p_1p_2q_1 + q_1^2p_2^2 + p_1^2 - q_2p_2^2 + q_2q_1^3 + 2q_2^2q_1$.) It will be observed that

[‡1] The "proof" of Lochak is incorrect since it allows no l-dependence in (3) and fails to precisely define ρ and $\tilde{\rho}$ (otherwise (6) for $n=2$ is a counter example.)

[‡2] A complementary class of integrable systems has been studied geometrically in ref. [9].

the integrals, or their combinations, in a h.s. system can always be ordered by where they enter η^2. Thus in Hénon–Heiles, H is the first integral and G the second.

Theorem. For a h.s. system:

(a) The principal balance corresponds to $\xi_1 \to$ infinity, $\xi_{i>1}$ a free constant and all flows H_i give rise to the same leading exponents $\{f_i, g_i\}$.

(b) All lower balances are present down to those with n free constants (excluding t_0) which are perforce just functions of the h_i.

(c) There are at least $L-4$ lower balances in which all $\{q_i, p_i\}$ diverge with the same $\{f_i, g_i\}$.

(d) If $\{q_i, p_i\}$ are weighted with the exponents in (c), then a perturbative expansion of the Hamilton–Jacobi equation for any $H_j(q, p)$ will yield bounds on the weighted degrees of all the $\{H_i\}$.

Remark 1. For the lowest hamiltonian only, the weighted degrees of H_i (with respect to the lowest balance on H_1) are bounded by just the Kowalevska resonances at the lowest balance.

Remark 2. Property (a) can be used to determine the degrees of $q(\xi)$ and for modest n the precise separating variable change can usually be found by inspection from the pole series.

Statements (a), (b) are proved by rewriting Hamilton's equations of motion for the ξ variables on the level set defined by the integrals. Part (c) follows from a topological argument based on Euler characteristics and Jacobi's theorem that the level set is essentially biholomorphic to $\gamma^{(n)}$. The action in (ξ, H) variables can be systematically expanded using the weights implied by (c) and then reexpanded in terms of $\{q_i\}$. From its form we can show how to organize the expansion of the Hamilton–Jacobi equation in q directly so as to obtain the degrees of h_i in η^2 and from them bounds on the weighted degrees of $H_i(q, p)$ [‡3].

Example. For $n=2$ Korteweg–de Vries, the lowest balance for H_1 is $f_i=(2, 4)$, $g_i=(5, 3)$ and the Kowalevska resonances are $\rho=(8, 10)$. For H_2, $f_i=(4, 3)$, $g_i=(10, 6)$ and the bounds on the degrees of H_i are $(16, 20)$.

Knowing that a problem is integrable in a certain way provides global information that permits features of the integrals to be deduced from the local information provided by the pole series and expansions of the Hamilton–Jacobi equation. If, however, only the Painlevé property is assumed, we are still able to plausibly construct a manifold on which global geometric methods may be applied; which to date have resulted in the concluding conjecture.

The manifold M is $2n$-complex-dimensional and augments the original phase space ($2n$ copies of the complex plane C) with the properties: (a) the differential equations are defined everywhere on M by polynomial hamiltonians, (b) ω^2 is preserved and extends to M, and (c) solutions exist for all times on M. The usual existence and uniqueness theorems for differential equations imply that distinct solutions remain distinct and that the time flow generates an analytic map of M 1 : 1 and onto itself. Apart from its abstract mathematical utility, the augmented manifold is precisely the construct one needs to numerically integrate Hamilton's equations of motion through all the singularities.

For each balance (1), we add a piece of surface to

[‡3] In spite of terminological similarities, our results have little in common with the work of Yoshida [13]. He shows that *if* a homogeneous polynomial integral exists, then its weighted degree occurs as a resonance in some balance. We claim that for a h.s. system bounds on the weighted degrees of all integrals can be found (cf. remark 1 to our theorem). The weights we assign do not in general yield homogeneous polynomials. Early in Yoshida's arguments (ref. [13], 2.8) it is incorrectly assumed that the scaling exponents which homogenize H_i become the $\{f_i, g_i\}$ in (1). This is true for H_1 in a h.s. system but not for $H_{i>1}$ where the desired exponents are fractional (e.g., for the second integral $G=H_2(q, p)$ in (5) the exponents $f_i=(-2/3, -2/3)$, $g_i=(-1, -1)$ appear to balance the t_2 equations and make H_i homogeneous, but there is no solution for the corresponding coefficients). This "phantom" solution may be interpreted (cf. comments on ref. [14]) as the lowest H_1 balance formally rewritten in t_2. It properly exists among the solutions to the Hamilton–Jacobi equation for H_i and furnishes the degrees of the other H_i. The pairing established by Yoshida (ref. [13], p. 376) obtains only for a resonance associated with a homogeneous integral. It is only at the lowest balance that all resonances correspond to integrals, but their conjugates then have $\rho \leqslant -1$ and except for t_0 do not occur in the series. Our eqs. (2), (3) encompass all the resonances at a principal balance and are the basis for the canonical variable change to the principal patch in M.

C^{2n} whose dimension m is the number of free constants excluding t_0. An open $2n$-dimensional coordinate patch is introduced to cover the new surface whose equation in local coordinates becomes just $0=u_1=...=u_{2n-m}$. Transition functions relate variables in the various coordinate patches when they overlap. They are constructed from a systematic expansion of the Hamilton–Jacobi equation which is equivalent to truncating the Laurent series (1) so as to include all the resonances. One then finds that the evolution equations in a given patch may themselves have poles which reproduce the next lower balances.

If one is able to build M, then one has proven that all the formal series (1) converge and established the distribution of poles in complex time. For these and reasons mentioned below, we advocate our Hamilton–Jacobi method that leads to M as the most informative way to restate and exploit the local Painlevé analysis.

The simplest illustration of the insights to be gained by constructing M is the Riccati equation [3] for $w(t)$ which is not hamiltonian or autonomous; however M may be constructed by inspection. Let

$$\mathrm{d}w/\mathrm{d}t=\sum_{i=0}^{2} a_i(t)w^i(t)\,. \qquad (7)$$

Observe that (7) has a simple pole $w\sim(t-t_0)^{-1}$, and that under the variable change $\bar{w}=w^{-1}$ (7) remains polynomial. We add to all $w\in\mathrm{C}$, the point at infinity $\bar{w}=0$ and use $w=\bar{w}^{-1}$ to glue all complex $\bar{w}\neq 0$ back onto the original phase space. All properties of M are satisfied. In this special case M is compact, so one can prove that all dependence of $w(t_2)$ on $w(t_1)$ is given by a fractional linear transformation [3].

This last remark is essentially nothing but a generalization of Liouville's theorem (analytic functions with algebraic growth at infinity are polynomials) applied to M. Further extensions of this reasoning in the concluding paragraphs below provide the fundamental connection between complex time properties and integrability.

We next construct M for a simplified version of the integrable Hénon–Heiles example (4). The principal balance is $f_i=(1,2)$, $g_i=(2,3)$. In addition to the energy h and t_0, there are two free constants $c_{1,2}$ defined by $q_1=c_1t^{-1}+\frac{1}{12}c_1^3t+\frac{1}{3}c_2t^2+...$ which are conjugate in the sense of (2). The resonances are $\rho_1=0$, $\rho_2=3$, and $\rho_h=6$. The only other balances are the two lowest ones previously noted, both with $f_i=(2,2)$ and $\rho_i=(6,8)$.

For the principal balance we add coordinates v_1, v_2, $v_3\in\mathrm{C}$, and u, also complex, but confirmed to a tube around $u=0$ which represents infinity. The transition functions for q, $p(u,\boldsymbol{v})$ are rational in u and polynomial in $\boldsymbol{v}$. They are compactly stated in terms of a generating function $A(\boldsymbol{q},v_2,v_3)$ $(q_1=x,\ q_2=-y)$,

$$A=\tfrac{4}{5}y^{5/2}+\tfrac{1}{2}x^2y^{1/2}-\tfrac{1}{32}x^4y^{-3/2}-v_2xy^{-1/2}-v_3y^{-1/2}\,, \qquad (8)$$

where $p_i=\delta A/\delta q_i$, $u=\delta A/\delta v_3$ and $v_1=\delta A/\delta v_2$. In the neighborhood of $u=0$, v_i tend to constants which approximate c_1, c_2 and h, while $\mathrm{d}u/\mathrm{d}t=1+\mathrm{O}(u^2)$. The transition functions are invertible near $u=0$ and $H(u,\boldsymbol{v})$ is polynomial and of course has the Painlevé property. The lowest balance series in (q,p) also appears as a pole in the principal patch according to $u\sim t$, $v_1\sim t^{-1}$, $v_2\sim t^{-4}$, $v_3\sim t^{-6}$. The transition functions for the lower balance patches will be given elsewhere.

On geometric grounds, G generates an analytic flow on the augmented manifold we constructed for H, which passes transversely through the hypersurface at infinity at almost all points. Therefore the G flow has the same principal balance exponents $\{f_i,g_i\}$ as H. There are two types of lowest balance. In the first $f_i=(2,6)$ and $g_i=(5,9)$ while the bounds on the degrees of H and G are (18, 24). At the other balance $f_{1,2}=-2$ and $g_{1,2}=1$ (i.e., the q tend to zero).

If we invoke the separability of (4), then the surface at infinity we must add to M to complete the flows on a level set is nothing but the separating curve (5). The principal balance patch covers all of γ except for a few points which are captured by the lowest balances. One can also verify that if the pole series for the principal balance of H were substituted into G, one would obtain precisely (5) with (ξ,η) replaced by $(\sqrt{2}c_1, c_2/\sqrt{2})$. This provides us with another rationale for the pairing between resonances, (3).

The projection of the G flow onto the curve at infinity is just the evolution one finds by treating (5) formally as a one degree of freedom hamiltonian. Observe that under (5), (ξ,η) blow up as $(s^{-1/3}, s^{-4/3})$ (or $v_1\sim t^{-1}$, $v_2\sim t^{-4}$ and $t\sim s^{1/3}$ as the lowest balance is approached from the principal patch). The fractional powers that arise from (5) also suggest an obvious way to explain the so-called weak Painlevé property of ref. [14]. Namely take a Painlevé system with one extra degree of freedom and express one of its other integrals as a function of the pole constants.

So far, we have only shown how to test for algebraic separability by providing enough information to calculate the integrals. This does not prove that algebraic integrals exist for an arbitrary polynomial Painlevé system. In fact they do not. The correct statement we conjecture is that *either* there exist functions that transform simply in time *or* the finite time map g_t on M is a fixed rational function. This precludes any sort of chaos since the variable time enters only the coefficients.

The Riccati equation exemplifies the second alternative. Similar conclusions follow whenever M can be sliced up into compact invariant submanifolds. The assumption of compactness can be relaxed to the requirement that g_t has a power law bound when its arguments tend to the boundary of M. When g_t depends on its arguments in an essential way, i.e., not polynomially and not simply in a combination which transforms trivially in time, then we believe that composition and the existence of g_t for all time leads to a contradiction.

For a general n-variable Painlevé system, with no lower balances, we can show that any two poles in complex time define a map from the hypersurface infinity to itself. This hypersurface is just C^{n-1} and all such maps commute. This family of maps, we believe, is sufficiently like the g_t one would obtain from a system of differential equations that stay entirely within C^{n-1}, that a full set of $n-1$ integrals should exist. The classical Painlevé transcendents have no lower balances and are integrable in this strong sense [1].

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