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# Dynamics of superfluid films

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A theory of the dynamical properties of a helium film near its superfluid transition is presented. Details are given of previously published results on the linear response of the film to a substrate oscillation. A key role is played by the diffusive motion of quantized vortices, which become free above the thermodynamic Kosterlitz-Thouless temperature $T_c$ but which only exist as bound pairs below $T_c$. An analogy with a two-dimensional plasma is presented and used. Contact is made with experiments involving oscillating substrates. The nucleation of single vortices from pairs is calculated, and this process is balanced against pair recombination to calculate the rate of decay of superflow below $T_c$. Formulas are worked out for the propagation and damping of third sound, and a discussion is given of hydrodynamic modes. An analogy between the dynamical equations for the film and Maxwell's equations is exploited.

## I. INTRODUCTION

### A. Purpose and outline

It has long been recognized that the decay of persistent currents as well as other dissipative processes in superfluids or superconductors are directly related to the rate at which vortices move across the flow path.[1–3] It has become clear, following the pioneering work of Kosterlitz and Thouless,[4–6] that in two-dimensional systems (i.e., in films thin compared to the superfluid correlation length), vortices are the key to understanding the transition between the superfluid and normal phases.[4–9] It is therefore natural to combine the Kosterlitz-Thouless analysis of the equilibrium distribution of interacting vortices with the earlier descriptions of vortex motion, in order to understand dynamic processes near the superfluid transition in thin films. A summary of the results of such an analysis has been given by the present authors in an earlier paper,[10] hereafter referred to as I. (See also the work of Doniach, Huberman, and Myerson.[11]) Applications to superconductors have also been discussed recently by several authors.[12–14]

In the present paper we shall give a more detailed derivation and discussion of the results previously presented in I. In addition to the application discussed there—the response of a film to an oscillating substrate, corresponding to the experiments of Bishop and Reppy[15]—we shall apply the theory to the propagation of third sound, and to the calculation of hydrodynamic modes above the transition temperature.

In outline, this paper is organized as follows. We first give a brief review of superfluidity in two dimensions, and discuss the philosophy underlying our calculations. In Secs. II–IV we apply the theory to situations, such as the oscillating substrate experiment, in which the supercurrent is divergence-free, and there are no macroscopic variations in the thickness and temperature of the film. In Sec. II, we consider the equations of motion for vortices and sketch an analogy with the diffusive motion of a system of two-dimensional charges. In Sec. III, we discuss the response of bound and free vortices to a small amplitude oscillation of the substrate (linear-response theory) in a form suitable for comparison with the experiments of Bishop and Reppy. Nonlinear behavior is discussed in Sec. IV, including the nucleation theory of the decay of superflow in the superfluid phase. This section includes discussion of the effects of vortex creation and annihilation at the boundary of the film.

In Sec. V, we generalize our discussion to the case where the divergence of the supercurrent is nonvanishing. We discuss the equations of motion in this case, and point out an analogy with the Maxwell equations for electrodynamics in two dimensions. We discuss here the applications to third-sound propagation and to the hydrodynamic modes above $T_c$. Section VI contains a summary of our key results.

Four related topics are relegated to appendices: the effects of interactions with an irregular substrate (A); some details and justifications for the dynamical theory (B); superfluid transition in films of finite thickness (C); and a compendium of relevant results from the static theory (D).

### B. Superfluidity in two dimensions

It is generally accepted that Bose condensation is the root cause of the anomalous properties of bulk liquid $^4$He near and below the $\lambda$ temperature $T_\lambda^{(3)}$. For a strongly interacting liquid like $^4$He, one usually means by Bose condensation that, at any temperature below $T_\lambda^{(3)}$, a three-dimensional sample of the liquid large on the scale of the bulk correlation length, and without hydrodynamic flows, may be characterized by a *macroscopic* order parameter $\psi$, a complex number related to the average value of the field operator $\psi_{\rm op}(\vec{r})$. Different values of the phase of this complex number correspond to different broken-symmetry states, in the same way as do different orientations of the macroscopic magnetization of a bulk ferromagnet. As the temperature is raised, fluctuations increase, the correlation length increases, and, finally, at $T_\lambda^{(3)}$, the order is destroyed. The modern theory[16] of this phase transition begins by asserting that the important statistical complexions whose excitation reduces the macroscopic order are states characterized by a spatially varying complex order parameter $\psi(\vec{r})$ defined on length scales down to a short-wavelength cutoff, which is much larger than the interparticle spacing but smaller than the correlation length. Each such complexion is assigned a free energy $F[\psi]$, which may be assumed for simplicity to have the Ginzburg-Landau form. The thermodynamic properties, which follow from doing the remaining partition sum as a functional integral over $\psi(r)$ of $e^{-F/k_BT}$, are believed to be insensitive to the precise form of the functional $F$. (For example, the same critical exponents are presumably obtained for a wide class of free-energy functionals $F$.)

In order to describe dynamical properties, one must consider an order parameter $\psi(\vec{r},t)$ which is a continuous function of time $t$ as well as of position $\vec{r}$. The time evolution of $\psi$ is described, in general, by stochastic equations of motion, which take into account interactions with fluctuations on the atomic scale, not included in the partially smoothed order parameter $\psi(\vec{r},t)$. Again, a detailed description of these equations of motion is not necessary for our purposes: superfluid hydrodynamics and dynamic critical properties are presumably independent of these details; identical results should be obtained for a wide class of models.

The concept of a spatially varying order parameter, which is needed to understand the critical statics and dynamics of bulk liquid $^4$He, becomes even more central in the case of thin films. There exists a rigorous proof[17] that in two or fewer dimensions macroscopic order in the sense of a nonzero space average of $\psi_{\rm op}(\vec{r})$ (for an infinite system) is incompatible with, and completely eliminated by, thermal fluctuations at any nonzero temperature. At the phenomenological level, the destruction of long-range order can be shown to be due to fluctuations in phase[18] of the local order parameter.

The importance of phase, as opposed to amplitude, fluctuations in two-dimensional systems is worth dwelling on further. In a bulk system near $T_\lambda^{(3)}$, amplitude and phase fluctuations are both significant. In a film, however, one is interested in two-dimensional transitions occurring at temperatures $T << T_\lambda^{(3)}$, which implies that amplitude fluctuations (about a deep minimum in the Ginzburg-Landau free energy) will be generally small. At the same time, a constant phase change costs no free energy, because of the degeneracy associated with the broken symmetry. Long-wavelength phase fluctuations are correspondingly cheap, even at temperatures below $T_\lambda^{(3)}$.

These arguments have been substantiated by a detailed renormalization-group analysis[19] in which it is shown that in two dimensions the stiffness against amplitude fluctuations becomes larger and larger as short-wavelength fluctuations are integrated out—i.e., small fluctuations in the magnitude of $\psi(r)$ are *irrelevant* —at the temperatures of interest for film transitions. Nevertheless, recognition that the smoothed order parameter $\psi(r)$ *can* pass through zero at isolated points is crucial to understanding the phase transition and critical phenomena of the two-dimensional superfluid. Zeros of the order parameter are necessary for the occurrence of vortices, which, as we have remarked, play the central role in the Kosterlitz-Thouless theory. Specifically, Kosterlitz and Thouless describe the transition from the superfluid to the normal state in terms of the unbinding of pairs of quantized vortices of opposite sign, moving in a sea of other vortex pairs. (Some remarks about the thermodynamic transition are contained in Appendix D.)

For the purposes of the present paper, we define a vortex as a point $\vec{r}_i$ where the real and imaginary parts of $\psi$ vanish. The circulation $\kappa_i$ about the vortex is then

$$\oint_C \vec{v}_s(\vec{r})\cdot d\vec{r} = \frac{2\pi\hbar}{m} n_i \ , \tag{1.1}$$

where $C$ is a contour enclosing the vortex, $n_i$ is an integer, $m$ is the mass of a helium atom, and $\vec{v}_s(\vec{r})$ is the "local superfluid velocity," related to the phase $\phi(r)$ of the order parameter by

$$\vec{v}_s(\vec{r}) = \frac{\hbar}{m}\vec{\nabla}\phi(r) = \frac{\hbar}{m}\frac{\mathrm{Im}(\psi^*\vec{\nabla}\psi)}{|\psi|^2} \ . \tag{1.2}$$

In practice, we need only consider vortices with $n_i = \pm 1$. The coefficient $(\hbar/m)$ in Eq. (1.2) is chosen so that $\vec{v}_s(\vec{r})$ transforms as a velocity under Galilean transformations of the system as a whole, i.e., simultaneous motion of the helium and substrate. [By its definition, $\psi(r)$ transforms like a one-particle wave function under Galilean transformations.] An argument showing that Eq. (1.2) retains its form in the case of relative motion of condensate and substrate is given in Appendix A.

If we neglect fluctuations in the magnitude of $\psi$, other than in the vortex cores, we can define a local supercurrent density.

$$\vec{j}_s(r) = \rho_s^0 \vec{v}_s(r) \quad , \tag{1.3}$$

where the coefficient $\rho_s^0$ is a "bare value" of the superfluid density, integrated across the thickness of the film. The value of $\rho_s^0$ is intended to include the effects of short-wavelength modulations of the substrate surface, as well as thermal excitations such as rotons and short-wavelength ripplons, but $\rho_s^0$ does not include the effects of vortices separated by distances greater than our microscopic cutoff.

If the average of $\vec{v}_s(\vec{r})$ in some macroscopic region deviates from zero, then there will be a net mass flow, equal to the average of $\vec{j}_s(\vec{r})$, which is simply the macroscopic supercurrent for $T < T_c$, or a "superfluid fluctuation contribution" to the mass current for $T > T_c$. The goal of the present paper is to estimate $\langle \vec{j}_s(\vec{r}) \rangle$ in various nonequilibrium situations.

As was mentioned earlier, the motion of vortices is the *sine qua non* for the decay of superflows below $T_c$. The kinetics of the motion of vortices thus becomes the central question for the dynamics of fluid flow. In Sec. II we shall introduce equations for the diffusion and net drift of vortices, which we assume to hold on the "macroscopic scale" of our initial short-wavelength cutoff. (As in the theories of three-dimensional hydrodynamics and dynamic critical phenomena, we assume that most details of the microscopic equations of motion are irrelevant for the long-wavelength behavior of interest.) In principle, we should then proceed to eliminate successively the vortex pairs of smallest separation, including the effects of the close pairs, by renormalization of the equations of motion of the remaining vortices of large separation. In practice, we have not carried out the renormalization calculation in detail. For the interactions between vortices we have simply used the results of the equilibrium renormalization anlaysis of Kosterlitz.[5,8] (See also the discussion of Young, in Ref. 9.) We have assumed that the Markovian form of the vortex equations of motion is preserved under the renormalization group, and that the diffusion constant $D$ is not drastically renormalized as one goes to the large distance scale. (An argument that there is no divergent renormalization of $D$ is presented in Appendix B.)

It is worthwhile to remark, by way of further justification for our approach to vortex interactions and motions, that there is an important difference between the behavior of vortices in two and three dimensions, as one approaches $T_c$ from below. In three dimensions, the core radius of a vortex (i.e., the "healing length" over which the order parameter rises from zero to its equilibrium mean value) is of the same order of magnitude as the correlation length, which diverges at $T_c$. Although a three-dimensional superfluid has a finite equilibrium density of vortex rings with ring diameters comparable to the correlation length, and although it is possible to argue that the three-dimensional phase transition may be understood as a divergence of the size of these rings, it does not seem reasonable to neglect the overlap of the cores of the vortices. In two dimensions, however, we take the vortex core diameter to remain finite as $T \to T_c$. [Although there is a length scale $\xi_-$ which diverges for $T \to T_c^-$, this length plays a relatively minor role in the two-dimensional system. (See Sec. III A, below.)] Furthermore, if bound vortex pairs at the atomic scale are ignored, the density of vortices is small relative to the finite core radius, even for temperatures slightly above $T_c$.

## II. DYNAMICAL THEORY

### A. Vortex motion

Consider a $^4$He film of uniform thickness, thin compared to a bulk correlation length, on a substrate which has a very large length $L$ in the $x$ direction and a large but finite width $W$ in the $y$ direction. The substrate is driven with a velocity $v_n(t)$. Our goal, in this and the next several sections, is to calculate the response of quantized vortices present in the film to this motion, the consequent change in the mean superfluid velocity of the film, and the effect of this response on the inertia and absorption of energy of the film.

To obtain the coupling between the vortex cores and the superfluid and substrate velocities, we adapt ideas of Hall and Vinen[20] for vortex lines in three dimensions. Motion of the vortex core relative to the local superfluid velocity leads to a Magnus force $\vec{F}_M$, whereas relative motion of the vortex core and the substrate leads to a drag force $\vec{F}_D$. We take these forces to have the forms

$$\vec{F}_M = n\rho_s^0 \frac{2\pi\hbar}{m} \hat{z} \times \left( \frac{d\vec{r}}{dt} - \vec{v}_s(\vec{r}) \right) \tag{2.1}$$

and

$$\vec{F}_D = B\left( \vec{v}_n - \frac{d\vec{r}}{dt} \right) + B' n\hat{z} \times \left( \vec{v}_n - \frac{d\vec{r}}{dt} \right) \quad . \tag{2.2}$$

Above, $\vec{r}$ is the position of a vortex core; $\rho_s^0$ is the superfluid density, integrated across the film thickness, which would be present in the absence of vortices; $\hat{z}$ is a unit vector in the $z$ direction; $B$ and $B'$ are phenomenological drag coefficients describing interactions with the substrate and with thermal excitations such as rotons and ripplons; $n$ $(=\pm 1)$ is the sign of the vortex; and the other symbols have been previously defined. The motion of a vortex will be determined by a balance between the forces described above and the thermal fluctuations associated with the interactions that cause the average drag force $\vec{F}_D$. The mean motion is obtained by setting $\vec{F}_M+\vec{F}_D=0$, which yields

$$\frac{d\vec{r}}{dt}=2\pi n\frac{D\hbar\rho_s^0}{mk_BT}\hat{z}\times[\vec{v}_n-\vec{v}_s(r)]+C[\vec{v}_n-\vec{v}_s(r)]+\vec{v}_s(\vec{r}) \ , \tag{2.3}$$

where

$$D=k_BT\frac{B}{[(2\pi\hbar/m)\rho_s^0-B']^2+B^2} \ , \tag{2.4}$$

$$C=1-\frac{[(2\pi\hbar/m)\rho_s^0-B'](2\pi\hbar/m)\rho_s^0}{[(2\pi\hbar/m)\rho_s^0-B']^2+B^2} \ . \tag{2.5}$$

We note that in the limit $B$, $B'\to 0$, we also have $C$, $D\to 0$, so that in the absence of drag forces the vortices would be carried along by the local superfluid velocity.

The temperature has been introduced in Eq. (2.3) in a physically motivated way. Considerations—given in Appendix B—of how thermal fluctuations must balance the dissipative effects contained in Eq. (2.3) (in order that the long-time average behavior of a system of vortices in a substrate at rest be consistent with thermal equilibrium) show that one must add on the right-hand side of this equation a fluctuating velocity whose autocorrelation fluctuation is proportional to $D$. With the inclusion of this fluctuating velocity we obtain our basic equations of motion for the positions $\vec{r}_i(t)$ of a collection of vortices:

$$\frac{d\vec{r}_i}{dt}=n_i\frac{D2\pi\hbar\rho_s^0}{mk_BT}\hat{z}\times(\vec{v}_n-\vec{v}_s^{\,i})+C(\vec{v}_n-\vec{v}_s^{\,i})+\vec{v}_s^{\,i}+\vec{\eta}_i(t) \ . \tag{2.6}$$

Here the index $i$ labels the vortices (of sign $n_i=\pm 1$), $\vec{v}_s^{\,i}$ is the local superfluid velocity at $\vec{r}_i$ (excluding the divergent self-field of the vortex at $\vec{r}_i$) and $\eta_i(t)$ are fluctuating Gaussian noise sources whose components $\eta_i^\alpha(t)$ satisfy

$$\langle\eta_i^\alpha(t)\eta_j^\beta(t')\rangle=2D\delta_{ij}\delta_{\alpha\beta}\delta(t-t') \ . \tag{2.7}$$

In order to provide a complete dynamical description, we must relate $\vec{v}_s$ to the positions of the vortices. In the present section, we shall restrict ourselves to situations where the superflow is divergence-free: $\vec{\nabla}\cdot\vec{v}_s=0$. Of course, we also have $\vec{\nabla}\times\vec{v}_s=0$, except at the vortex points. We require, in addition, that the normal component of $\vec{v}_s$ vanish at the edges of the film, and that $\vec{v}_s$ have the correct circulation about each vortex point.

The solution to the above equations can be written in the form

$$\vec{v}_s(r)=\vec{u}_s+\left(\frac{\hbar}{m}\right)\hat{z}\times\sum_j n_j\vec{\nabla}G(\vec{r},\vec{r}_j) \ , \tag{2.8}$$

where the symbols not previously defined are $\vec{u}_s$ the average of $\vec{v}_s(r)$ over the film area and $G$, a Green's function describing the flow field (obeying the correct boundary condition at the film edges) due to a positive vortex at $\vec{r}_j$. Thus $G$ is the solution of

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)G(\vec{r},\vec{r}_j)=2\pi\Delta(\vec{r}-\vec{r}_j) \ , \tag{2.9}$$

where $\Delta(\vec{r}-\vec{r}_j)$ is concentrated for $\vec{r}$ in a region on the scale $a_0$ of the vortex core near $\vec{r}_j$ and $\int d^2r\,\Delta(r)=1$. The boundary condition of no flow out of the film is satisfied by requiring $G(\vec{r},\vec{r}_j)=0$ for $\vec{r}$ on the edges. With this choice of $G$, the average over the film of the last term in Eq. (2.8) is seen to be zero, so that the average of $\vec{v}_s(r)$ is indeed $\vec{u}_s$. Far from the edges and the core region $G\approx\ln(|r-r_j|/a_0)$, where $a_0$ is a vortex core diameter.

The time dependence of the average superfluid velocity $\vec{u}_s(t)$ produced by the motion of vortices is given by

$$\frac{d\vec{u}_s}{dt}=-\sum_i\frac{2\pi\hbar}{mLW}n_i\hat{z}\times\frac{d\vec{r}_i}{dt} \ . \tag{2.10}$$

To understand Eq. (2.10), note that the displacement of a positive vortex by an amount $dy$ in the $y$ direction changes the average velocity in the $x$ direction in a strip of width $dy$ from $-(\pi\hbar/mL)$ to $+(\pi\hbar/mL)$. Equation (2.10) is a microscopoic restatement of the equation for the decay of two-dimensional superflow due to vortex motion, as discussed by Langer and Reppy.[3] Similar ideas were used earlier by Anderson.[1]

Equations (2.6), (2.8), and (2.10) are our basic dynamical equations. These equations are considerably illuminated by an analogy with the dynamics of a two-dimensional plasma, confined between capacitor plates, and subjected to an oscillating electric field, in which charges move by diffusion. We develop this analogy in Sec. II B.

### B. Analogy with diffusive motion of electric charges in two dimensions

The plasma analog of the system described above is the following. We imagine two-dimensional charges moving in the $xy$ plane.[21] Let the motion be confined to the region between two capacitor plates (lines in our two-dimensional space), parallel to the $x$ axis and of large length $L$. Let $W$ be the distance separating the plates. We take the charges to have the values $nq_0(n=\pm 1)$, where $q_0$ will be specified below. The Langevin equation for the diffusive motion of a charge $n_i q_0$ at $\vec{r}_i$ is

$$\frac{d\vec{r}_i}{dt}=\frac{D}{k_B T}n_i q_0 \vec{E}(\vec{r}_i)+\vec{\eta}_i \ , \qquad (2.11)$$

where $\vec{E}(\vec{r}_i)$ is the electric field at $\vec{r}_i$, excluding the divergent self-field of the charge at $\vec{r}_i$. The mobility of the charge has been written as $(D/k_B T)$ where $D$ is the diffusion constant and $T$ is the temperature of the system. The fluctuating velocity $\vec{\eta}_i$, which satisfies Eq. (2.7), may be considered to arise from interactions with a heat bath at temperature $T$. The local electric field $\vec{E}(\vec{r})$ is determined by the equation

$$\vec{E}(\vec{r})=\langle E\rangle+2q_0\sum_j n_j \vec{\nabla} G(\vec{r},\vec{r}_j) \ , \qquad (2.12)$$

where $G(\vec{r},\vec{r}_j)$ is minus one-half the electrostatic potential due to unit charge at $\vec{r}_j$ when the capacitor plates are grounded. $G$ then obeys Eq. (2.9) and the boundary condition discussed below that equation, namely, $G(\vec{r},\vec{r}_j)=0$ for $\vec{r}$ at the plates. The average electric field $\langle E\rangle$ is related to the external field $\vec{E}_{\text{ext}}$, due to charging the capacitor plates, as follows:

$$\langle E\rangle=\vec{E}_{\text{ext}}-4\pi\vec{P} \ , \qquad (2.13)$$

where $\vec{P}$ is the average dipole moment per unit area of the system of charges. The rate of change of the average dipole moment due to the motion of the charges is

$$\frac{d\vec{P}}{dt}=\sum_j \frac{n_j q_0}{LW}\frac{d\vec{r}_j}{dt} \ . \qquad (2.14)$$

Note the analogy between the pairs of Eqs. (2.6) and (2.11); (2.8) and (2.12); and (2.10) and (2.14). The only difference in form is contained in the second and third terms on the right of Eq. (2.6), hereafter called the "convective" terms. These terms carry an isolated vortex in the direction $\vec{v}_n$ and leave unchanged the separation vector of an isolated pair. Thus the convective motion of single vortices and pairs does not contribute to $d\vec{u}_s/dt$ in Eq. (2.10).

If we ignore these terms we obtain the transcription

$$q_0\vec{E}(\vec{r}) \Leftrightarrow (2\pi\rho_s^0\hbar/m)\hat{z}\times[\vec{v}_n-\vec{v}_s(\vec{r})] \ ,$$

$$4\pi q_0\vec{P} \Leftrightarrow (2\pi\rho_s^0\hbar/m)\hat{z}\times\vec{u}_s \ ,$$

and

$$q_0\vec{E}_{\text{ext}} \Leftrightarrow (2\pi\rho_s^0\hbar/m)\hat{z}\times\vec{v}_n \ .$$

Furthermore, by comparing the interaction energy of a pair of charges and a pair of vortices, one obtains the correspondence $2q_0^2 \Leftrightarrow 2\pi\rho_s^0(\hbar/m)^2$.

The motion of vortices in a helium film on an oscillating substrate is thus equivalent to the diffusive motion of two-dimensional charges between capacitor plates subjected to an oscillating potential difference. This analogy will prove very useful in calculating the inertia and absorption of energy of the system. We turn to such calculations in Sec. III A. We shall use the language of charges or vortices interchangeably, the equivalences given in the last paragraph always being understood.

## III. LINEAR RESPONSE

### A. Dynamical dielectric constant

For small substrate velocities at a finite frequency $\omega$, the inertial and dissipative properties of the helium film may be related to a dynamical dielectric constant defined as follows:

$$\vec{u}_s(\omega)=[1-\epsilon^{-1}(\omega)]\vec{v}_n(\omega) \ , \qquad (3.1)$$

where $\vec{u}_s$ is the space average of $\vec{v}_s(\vec{r})$, as in Eq. (2.8). In the plasma analog, this equation is the conventional relation

$$4\pi\vec{P}(\omega)=[1-\epsilon^{-1}(\omega)]\vec{E}_{\text{ext}}(\omega) \ . \qquad (3.2)$$

The momentum per unit area of film is given by

$$\vec{g}(\vec{r})=\rho_s^0\vec{v}_s(\vec{r})+(\rho-\rho_s^0)\vec{v}_n \ . \qquad (3.3)$$

The reaction force on the substrate due to the film, per unit area, is equal to $-d\vec{g}/dt$, and the power dissipated per unit film area is the time average of $\vec{v}_n\cdot d\vec{g}/dt$, namely,

$$P_0=\tfrac{1}{2}\rho_s^0(v_n^{\max})^2\omega\,\text{Im}[-\epsilon^{-1}(\omega)] \ . \qquad (3.4)$$

If the substrate is part of a simple harmonic oscillator with mass $M$ much greater than the mass of the helium film, the stored energy is $\frac{1}{2}M(v_n^{\max})^2$. If there are no losses other than the power dissipated in the film, then the $Q$ of the system is given by

$$Q^{-1}=\frac{\rho_s^0 A}{M}\text{Im}[-\epsilon^{-1}(\omega)] \ , \qquad (3.5)$$

where $A$ is the surface area of the substrate. The force exerted by the film on the substrate also leads to a shift $\Delta P$ in the period $P$ of the harmonic oscillator given by

$$\frac{2\Delta P}{P}=\frac{A\rho_s^0}{M}\text{Re}[\epsilon^{-1}(\omega)] \ . \qquad (3.6)$$

In attempting to calculate $\epsilon(\omega)$, different regimes of temperature and frequency must be distinguished.

From a conceptual point of view, the most straightforward regime corresponds to temperatures below the Kosterlitz-Thouless temperature $T_c$ and all frequencies $\omega$ such that $\omega < D/a_0^2$. We proceed to sketch the calculation for $\epsilon(\omega)$ in this regime. According to the Kosterlitz-Thouless theory, the equilibrium thermodynamics below $T_c$ is described by a dilute gas of pairs of renormalized vortices. The dynamics of an *isolated* pair is obtained by subtracting the Langevin equations for two vortices of opposite sign moving in their mutual velocity fields and subject to an oscillating external driving force. Working in this way with Eq. (2.6) or (2.11), we find the following equation of motion for the relative coordinate $\vec{r}$:

$$\frac{d\vec{r}}{dt} = -\frac{2D}{k_B T}\frac{\partial U}{\partial \vec{r}} + \vec{\eta}(t) \quad . \tag{3.7}$$

In the plasma language of Sec. II B

$$U(\vec{r}) = 2q_0^2 \ln r - q_0 \delta\vec{E}\cdot\vec{r} - 2\mu_0 \ ,$$

where $\mu_0$ is defined in Appendix D, and $\delta\vec{E}$ is a small macroscopic field, which we take to vary sinusoidally in time with circular frequency $\omega$. The noise source $\vec{\eta}(t)$ obeys $\langle \eta^\alpha(t)\eta^\beta(t')\rangle = 4D\delta(t-t')\delta_{\alpha\beta}$. It may be worth emphasizing that the convective terms in Eq. (2.6), i.e., the second and third terms on the right, cancel between the two members of the pair and do not enter Eq. (3.7) for the relative motion.

As in the Kosterlitz-Thouless theory, which we wish our dynamic theory of pairs to approach in the static limit, we include the effect of smaller pairs on the pair singled out in Eq. (3.7) by a screening correction to the interaction energy. This then becomes

$$U(\vec{r}) = 2q_0^2 \int_{a_0}^{r} \frac{dr'}{r'\tilde{\epsilon}(r')} - q_0\delta\vec{E}\cdot\vec{r} - 2\mu_0 \quad . \tag{3.8}$$

Here $\tilde{\epsilon}(r)$ is Kosterlitz's static length-dependent dielectric constant. We use a *static* dielectric function in Eq. (3.8) because the screening is produced by pairs of small separation, which relax more rapidly than the larger pair being considered in Eq. (3.7). The properties of $\tilde{G}(r)$ are reviewed in Appendix D, below.

In Ref. 22, Eqs. (3.7) and (3.8) were used to calculate—via a Fokker-Planck[23] equation—the time-dependent polarization per unit area, to linear order in $\delta\vec{E}$. Without further approximation, it was shown that the resulting bound-pair contribution to the dynamical dielectric constant has the form

$$\epsilon_b(\omega) = 1 + \int_{a_0}^{\infty} dr \, \frac{d\tilde{\epsilon}}{dr} g(r,\omega) \ , \tag{3.9}$$

which form was earlier written down[10] on intuitive grounds. The function $g(r,\omega)$ is a response function for pairs of separation $r$; $g(r,\omega) = 1$ corresponds to local equilibrium. There are two scales of length that enter this response function: the coherence length $\xi_-$, defined in Refs. 10 and 22 and in Appendix D, and the diffusion length $r_D \equiv (2D/\omega)^{1/2}$. If $\xi_- \ll r_D$, as is true for the frequencies at which experiments have been done for essentially all $T < T_c$ (see Sec. III B), then one finds[22] to a rather good approximation that

$$g(r,\omega) \approx 14Dr^{-2}/(14Dr^{-2} - i\omega) \quad . \tag{3.10}$$

The physics of Eq. (3.10) is simply that the diffusion length determines the crossover between the smaller pairs which can and the larger pairs which cannot equilibrate to the oscillating field. Under the assumed conditions the integral (3.9) may then be well estimated by noting that in the range of separations $r$ for which $g(r,\omega)$ passes from its low to its high-frequency behavior, the dielectric constant $\tilde{\epsilon}(r)$ is a slowly varying function of $\ln r$. We may thus approximate

$$\mathrm{Re}\, g(r,\omega) \approx \theta(14Dr^{-2} - \omega) \ , \tag{3.11}$$

$$\mathrm{Im}\, g(r,\omega) \approx \tfrac{1}{4}\pi r \delta(r - \sqrt{14D/\omega}) \ , \tag{3.12}$$

where $\theta(x) = 1$, for $x > 0$ and $\theta(x) = 0$, for $x < 0$. It then follows that, for $T \leq T_c$, $\epsilon(\omega) = \epsilon_b(\omega)$, where

$$\mathrm{Re}\,\epsilon_b(\omega) = \tilde{\epsilon}(r = (14D/\omega)^{1/2}) \tag{3.13}$$

and

$$\mathrm{Im}\,\epsilon_b(\omega) = \tfrac{1}{4}\pi \left[ r\left(\frac{d\tilde{\epsilon}}{dr}\right)\right]\Bigg|_{r=(14D/\omega)^{1/2}} \ . \tag{3.14}$$

Eqs. (3.4), (3.5), (3.13), and (3.14) are the principal results of our theory below $T_c$ and give the inertia and dissipation of weakly driven films in this region. The restriction $\omega < D/a_0^2$ mentioned below Eq. (3.6) is actually not very restrictive at all (see Sec. III B). It means simply that for the theory to apply, the frequency should not probe microscopic lengths where the idea of a low density of renormalized pairs is inapplicable.

Note that nothing dramatic happens to $\epsilon_b(\omega)$ as $T$ approaches $T_c$ from below. The point is that the thermodynamic $T_c$ is the temperature at which pairs of *infinite* separation unbind. On smaller length scales even above $T_c$, the Kosterlitz recursion relations[5] show that the charged system still behaves like an insulator. Above $T_c$, there is a correlation length $\xi_+(T)$ which determines the length scale above which it is not meaningful to speak of pairs; it is then more accurate to associate one free charge (vortex) per $(\xi_+)^2$ of film area. Thus above $T_c$ we must modify Eq. (3.9) to read

$$\epsilon_b^+(\omega) = 1 + \int_{a_0}^{\xi_+} dr \frac{d\tilde{\epsilon}}{dr} g(r,\omega) \tag{3.15}$$

and add to this bound-pair effect the contribution of "free vortices." Eqs. (3.10)–(3.12) for $g(r,\omega)$ remain valid when $r_D << \xi_+$. The region of temperatures for which $\xi_+ \approx r_D$ is more difficult to deal with, but we find in this region that the response is dominated by free vortices. It therefore suffices for practical purposes to retain Eqs. (3.10)–(3.12) above $T_c$.

As far as the free charges are concerned, we make a "Debye-Hückel" approximation, in which each charge diffuses in the *macroscopic* electric field. We thus obtain a contribution $\vec{P}_f$ to the dipole moment per unit area, given by

$$\frac{d\vec{P}_f}{dt} = n_f q_0^2 \frac{D}{k_B T} \langle \vec{E} \rangle \ , \tag{3.16}$$

where $n_f$ is the density of free charges ($\sim \xi_+^{-2}$). Adding this contribution to the dipole moment of the bound pairs we then get for $T > T_c$,

$$\epsilon(\omega) = \epsilon_b^+(\omega) + 4\pi i \sigma/\omega \ , \tag{3.17}$$

where $\sigma$ is the electrical conductivity in the plasma analogy,

$$\sigma = n_f q_0^2 D / k_B T \ . \tag{3.18}$$

Returning to the language of vortices, we set $q_0^2 \rightarrow \pi\hbar^2 \rho_s^0/m^2$. The formula for $\epsilon(\omega)$ may then be used in Eqs. (3.4) and (3.5) to derive observable consequences.

It should be remarked that the quantities $\xi_+$, $n_f$, etc., have only been defined, so far, up to numerical factors of order unity. This lack of precision is not serious for our purposes, since we will not attempt to calculate the absolute values of dynamic quantities. Nevertheless, it is possible to define a number of correlation lengths in a precise way and then discuss the differences between these lengths.

In the context of the superfluid or $XY$ model, it is natural to define a correlation length for the order-parameter correlation function from the requirement

$$\langle \psi^*(r)\psi(0) \rangle \sim e^{-r/\tilde{\xi}_+} \ , \tag{3.19}$$

for $r \rightarrow \infty$. One may also define a screening length $k_s^{-1}$ from the decay at large distances of the vortex-charge correlation function

$$\langle N(\vec{r}) N(0) \rangle \sim \exp(-k_s r) \ , \tag{3.20}$$

where $N(r)$ is the vortex-charge density defined in Eq. (5.3), below. There is reason to believe[24] that $k_s^{-1}$ and $\tilde{\xi}_+$ differ by precisely a factor of 2,

$$k_s = 2\tilde{\xi}_+^{-1} \ . \tag{3.21}$$

In the Debye-Hückel theory, the relation between $k_s$ and $n_f$ is

$$k_s^2 = \frac{4\pi q_0^2 n_f}{\epsilon_b k_B T} \ . \tag{3.22}$$

Clearly, there is not really a sharp line between free charges and bound charges, so that $\epsilon_b$ is not precisely defined, and even within the Debye-Hückel theory there is some arbitrariness in the definition of $n_f$. If we simply identify $\epsilon_b$ with $\epsilon_c$, the dielectric constant at $T_c$, and we use the Kosterlitz-Thouless result $q_0^2/\epsilon_c k_B T_c = 2$, we obtain from Eqs. (3.21) and (3.22)

$$n_f \approx \frac{k_s^2}{8\pi} = \frac{\tilde{\xi}_+^{-2}}{2\pi} \ . \tag{3.23}$$

### B. Comparison with experiment

A detailed comparison of the linear-response theory of Sec. III A and the oscillating substrate experiment has been published together with the description of the experiments.[15,25] We therefore confine ourselves here to a qualitative discussion of how the theory explains one striking feature of the observations, namely the peak in the dissipation at fixed frequency as the temperature is raised.

The basic equations for making correspondence between theory and this feature of the experiment are Eq. (3.5) for the dissipation and Eqs. (3.9), (3.15), and (3.17) for the frequency-dependent dielectric constant.

From these equations, one sees that the bound-pair contribution to the imaginary part of the dielectric constant is given by

$$(\rho_s^0)^{-1} \operatorname{Im}\epsilon_b(\omega) = \frac{\hbar^2}{m^2 k_B T} \pi^4 y^2 \left( l = \tfrac{1}{2} \ln \frac{14D}{a_0^2 \omega} \right) \ , \tag{3.24}$$

where we have used Eq. (3.14) and expressed the derivative of the length-dependent static dielectric constant in terms of the length-dependent activity via the Kosterlitz recursion relations.[5,9] (See also, Appendix D.) Since $y^2(l)$ measures the number of pairs with separation $a_0 e^l$, Eq. (3.24) says very physically that the imaginary part is proportional to the number of pairs (on the length scale determined by the frequency and the diffusion constant) whose motion is maximally out of phase with the driving force. Now, as one raises the temperature at fixed $l$, $y^2$ grows smoothly, reaching the value $(4\pi l)^{-1}$ at $T = T_c$ and continuing to grow, without any discontinuity at $T_c$, until the bound-pair contribution is cut off by the upper limit in Eq. (3.15). This is the origin of the low-temperature side of the peak. At the high-temperature end, the peak is due to the motion of free vortices. Here the imaginary part is given by the second term of Eq. (3.17). Since the number of free vortices, proportional to $(\xi_+)^{-2}$, grows as temperature increases, this imaginary part dominates all other terms at high temperatures, and one obtains from Eqs. (3.5) and (3.18) the result

$$Q^{-1} \rightarrow \frac{A}{M} \left( \frac{m}{2\pi\hbar} \right)^2 \left( \frac{k_B T}{D} \right) \left( \frac{\xi_+^2}{F} \right) \omega \ . \tag{3.25}$$

Here we have written $n_f = (F/\xi_+^2)$, with $F$ a coefficient of proportionality which we expect to be of order unity. Now $\xi_+ \approx a_0 \exp[2\pi/x(T)]$, where $x(T) = b(|1 - T/T_c|)^{1/2}$ and $b$ is a nonuniversal constant.[5,10] The growth of $\xi_+$ as the temperature is lowered produces the increase in dissipation. This is the explanation of the high-temperature end of the peak in the dissipation.

The fit of theory and experiment involves the adjustment of several parameters—explained in detail in Ref. 25. It is therefore important to emphasize that a good correspondence is obtained with reasonable values for the parameters. In particular, the main parameter of the dynamical theory, the diffusioin constant $D$ has to be assigned a value of the order of $10^{-4}$ cm$^2$/sec. This is consistent with our idea that $D$ is a microscopic quantity not significantly renormalized by the transition to larger length scales. (See Sec. I and Appendix B.) At the microscopic level we estimate $D$ by dimensional arguments (Appendix B) to be of order $\hbar/m \sim 10^{-4}$ cm$^2$/sec. As explained in Ref. 25, the other parameters in the fit also take on reasonable values. We also remark in passing that with this value of $D$ the diffusion length $r_D \equiv (2D/\omega)^{1/2} \sim 10^{-4}$ cm for the experimental frequency $\sim$1 KHz. Thus $r_D$ is indeed much larger than the core radius $a_0 \sim 10^{-8}$ cm, as assumed in Sec. VI.

A more stringent test of the theory is possible if the experiment is done at several frequencies on the same film. If the basic ideas of the theory are correct, the dissipation peak, and the corresponding drop in the superfluid density should move to higher temperatures as the frequency is increased. The reason is, of course, that one then shortens the diffusion length and thus probes the system on shorter length scales which soften at higher temperatures.

## IV. NONLINEAR EFFECTS

### A. Theory of nucleation

The linear theory of Sec. III A is based on assumptions that are violated under easily obtainable experimental conditions.[26] In the present section we shall consider the effects of large amplitude, low-frequency substrate motion for $T < T_c$. Specifically, we shall calculate the rate of creation of free vortices from pairs in response to a finite amplitude substrate oscillation, when the frequency is small in the sense that $r_D[\equiv (D/\omega)]^{1/2} > r_c$, where $r_c$ is a critical nucleation radius discussed below.

We start from Eqs. (3.7) and (3.8). From these equations, one obtains in a standard way a Fokker-Planck equation[23]:

$$\frac{\partial \Gamma}{\partial t}(\vec{r}) = -\frac{\partial}{\partial \vec{r}}\left[-2D \exp\left(\frac{-U}{k_B T}\right)\frac{\partial}{\partial \vec{r}}\Gamma(\vec{r})\exp\left(\frac{U}{k_B T}\right)\right], \tag{4.1}$$

where $\Gamma(\vec{r})$ is the number of pairs per unit film area per unit area of pair separation in the neighborhood of the separation $\vec{r}$.

In a finite field $\vec{E}$, which we take to point in the $x$ direction, the potential $U(\vec{r})$ of Eq. (3.8) has a saddle point at $\vec{r}_c = (r_c, 0)$ where $r_c$ is determined by the equation $E r_c \tilde{\epsilon}(r_c) = 2q_0$. (See Fig. 1.)

To calculate the rate of nucleation of free vortices from bound pairs, we follow the standard method of seeking a stationary solution of Eq. (4.1) in which there is a steady current of pairs passing over the saddle point shown in Fig. 1. Making an expansion of $U(\vec{r})$ in this neighborhood, and defining $U(r_c, 0) \equiv U_c$, we obtain

$$\begin{aligned}\exp(-U/k_B T) \approx \exp(-U_c/k_B T) \\ \times \exp[q_0^2(\delta x)^2/k_B T\tilde{\epsilon}(r_c) r_c^2] \\ \times \exp[-q_0^2(\delta y)^2/k_B T\tilde{\epsilon}(r_c) r_c^2] .\end{aligned} \tag{4.2}$$

Here $\delta x$ and $\delta y$ are $x$ and $y$ deviations from $(r_c, 0)$. Using the scale dependent stiffness constant and activity—see Appendix D—Eq. (4.2) may be expressed as

$$\begin{aligned}\exp\left(\frac{-U}{k_B T}\right) \approx \frac{a_0^4}{r_c^4} y^2(l_c)\exp[2\pi K(l_c)] \\ \times \exp\left(\frac{\pi K(l_c)(\delta x)^2}{r_c^2}\right) \\ \times \exp\left(\frac{-\pi K(l_c)(\delta y)^2}{r_c^2}\right) ,\end{aligned} \tag{4.3a}$$

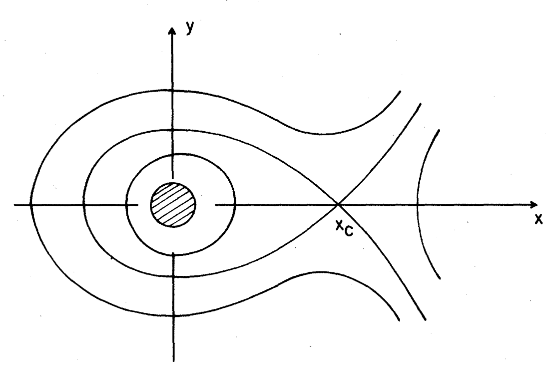


FIG. 1. Contours of constant potential between a pair of oppositely charged vortices at separation $\vec{r} = (x,y)$. The potential decreases upon approaching origin, but has a hard core (shown cross hatched in the figure) of radius $a_0$. A uniform superfluid velocity has been imposed in the $y$ direction. There is a saddle point at $\vec{r}_c = (x_c, 0)$, over which the pair separation can "escape" to large positive $x$.

where

$$l_c \equiv \ln(r_c/a_0) \quad . \tag{4.3b}$$

The vector in square brackets in Eq. (4.1), call it $\vec{j}(\vec{r})$, is a current denstiy. With the normalization of $\Gamma$ given below Eq. (4.1), the quantity $R \equiv \int d(\delta y) \times j_x(r_c, \delta y)$ has the interpretation of the number of separating pairs, i.e., the number of pairs in which the internal separation crosses the saddle point, per unit film area per unit time. Within the Gaussian approximation of Eqs. (4.1), (4.2), and (4.3), the required steady-state solution in the vicinity of the saddle point $\vec{r}_c \approx (r_c, 0)$ is found by standard methods.[27] We make the ansatz that

$$j_y = 0 \quad , \tag{4.4a}$$

$$j_x(\delta y) = R\,[K(l_c)/r_c^2]^{1/2} \times \exp[-\pi K(l_c)(\delta y)^2/r_c^2] \quad . \tag{4.4b}$$

Note that Eq. (4.4a) together with the steady-state condition $\vec{\nabla} \cdot \vec{j} = 0$ implies that $j_x$ is independent of $\delta x$. It then follows from Eqs. (4.1) and (4.3) that

$$R \left[\frac{K(l_c)}{r_c^2}\right]^{1/2} \int_{\delta x}^{\infty} d(\delta x') \exp\left[\frac{-\pi K(l_c)(\delta x')^2}{r_c^2}\right] = (2D)\left[\frac{a_0}{r_c}\right]^4 y^2(l_c) \exp[2\pi K(l_c)]\Gamma(\delta x, \delta y)\exp(U/k_B T) \quad . \tag{4.5}$$

The assumption that $j_y = 0$ near the saddle point can be satisfied by choosing $\Gamma$ to have the same $\delta y$ dependence as $\exp(-U/k_B T)$ in this neighborhood. Evaluating Eq. (4.5) for $\delta x < -[r_c^2/2\pi K(l_c)]^{1/2}$, where $\Gamma$ has its equilibrium value $a_0^{-4}\exp(-U/k_B T)$, gives the principal result of this section

$$R = (2D)[y^2(l_c)/r_c^4]\exp[2\pi K(l_c)] \quad . \tag{4.6}$$

Before proceeding to interpret this result, we must make some comments about the approximations made. The calculation given above, based on Gaussian approximations, applies strictly only for the situation of rare escapes over a high barrier. The condition $2\pi K >> 1$ is thus implicit. In fact, for our problem $2\pi K \approx 4$. Thus we cannot be sure that Eq. (4.6) is not missing a numerical factor, but there is no reason to doubt that the $r_c$ dependence is correct. The situation is quite different in some other two-dimensional barrier escape problems, e.g., a potential of the form $U(r) = -A[\frac{1}{2}r^{-2} + (x/r_c^3)]$ discussed by Onsager and McCauley.[28] In this case, the angle subtended by the region of small barrier becomes large as $r_c \to \infty$ so that the saddle-point method becomes meaningless.

Returning to Eq. (4.6), we remark that there are two length scales in the problem: $r_c$, and $\xi_-$—the length which determines when $\tilde{\epsilon}(r)$ approaches its finite asymptotic value at the temperatures below $T_c$ which we are considering. Both these lengths have to be macroscopic, i.e., much greater than $a_0$, for the derivation given above to be valid. However the result (4.6) depends on the ratio of these two large lengths.

For $r_c >> \xi_-(T)$ we have[5,10,25] (see also Appendix D)

$$4\pi y(l_c) = x(T)\exp\{-\tfrac{1}{2}[x(T)l_c]\} \quad , \tag{4.7a}$$

$$K(l_c) = \frac{2}{\pi}[1 + \tfrac{1}{4}x(T)] \quad , \tag{4.7b}$$

where $x(T) = b(|1 - T/T_c|)^{1/2}$ determines $\xi_-$ via $\ln(\xi_-/a_0) = [x(T)]^{-1}$. In this limit, which is violated very close to $T_c$ where $\xi_- \to \infty$, we obtain, by substituting Eq. (4.7) into Eq. (4.6), a formula given in I[29]:

$$R \propto D a_0^{-4}[x(T)]^2(a_0/r_c)^{4+x(T)} \quad . \tag{4.8}$$

For the vortex-pair problem, the length $r_c$ is obtained, by using the definition below Eq. (4.1) and the dictionary in Sec. II B, to be

$$r_c = \hbar/m|v_n - u_s|\tilde{\epsilon}(r_c) \quad . \tag{4.9}$$

For fixed $v_n$, as the temperature is increased towards $T_c$, the rate (4.8) increases. Finally the condition $\xi_- << r_c$ is violated and we must use approximations valid in the limit $x(T)l_c << 1$, namely,

$$4\pi y(l_c) = l_c^{-1} \quad , \tag{4.10a}$$

$$K(l_c) = \frac{2}{\pi}(1 + \tfrac{1}{2}l_c^{-1}) \quad . \tag{4.10b}$$

The rate (4.8) then has the form

$$R\ (T \to T_c^-) \propto \frac{D}{r_c^4}(l_c)^{-2} \quad . \tag{4.10c}$$

### B. Boundary effects and decay of superflow

In order to calculate the density of free vortices $n_f$, one must combine the rate of nucleation of free vortices calculated above with the rate of annihilation via the inverse process, in which two free vortices of opposite sign collide and form a new bound pair. In addition, for a sample of finite width, one must consider the annihilation and generation of vortices at film edges.

In vortex nucleation theories of superflow decay in films,[2,3] it has often been assumed that bulk generation of free vortices is balanced by annihilation at the boundaries. Although annihilation at boundaries is certainly important in sufficiently narrow samples, we shall argue that this process is balanced by an enhanced rate of creation at free vortices near the boundary. For samples sufficiently wide that boundary creation is negligible, one may also neglect boundary annihilation, and the steady-state vortex density is limited by pair recombination in the bulk of the film. The steady-state density $n_f$ will turn out to be essentially independent of whether bulk or edge effects are dominant.

To study this point further, we calculate $R_w$, the rate of production of free vortices at the edge of the film, by calculating the rate at which a vortex can escape from its image charge at the film boundary. Consider for simplicity an equilibrium population of vortices bound to a straight film edge of length $L$ parallel to the $x$ axis, when there is a uniform superflow $\vec{u}_s - \vec{v}_n$ in the $y$ direction (see Fig. 2). Under these conditions, an isolated vortex feels a wall potential $U_w$ which is only a function of $x$, the distance to the wall. In the Coulomb gas language, one finds

$$U_w(x) = q_0^2 \int_{a_0}^{x} \frac{dx'}{x'\tilde{\epsilon}(2x')} - q_0 \delta E x - \mu_0 \tag{4.11}$$

in contrast to Eq. (3.8). The wall potential assumes half the value corresponding to a *pair* of separation $2x$, since the region $x<0$ occupied by the fictitious image charge does not contribute.

The Fokker-Planck equation for $\Gamma_w(x)$, the density of vortices within $r_c$ of the wall, is now

$$\frac{\partial \Gamma_w}{\partial t} = \frac{-\partial}{\partial x}\left[ -D \exp\left(\frac{-U_w}{k_B T}\right) \frac{\partial}{\partial x} \Gamma_w \exp\left(\frac{U_w}{k_B T}\right)\right] . \tag{4.12}$$

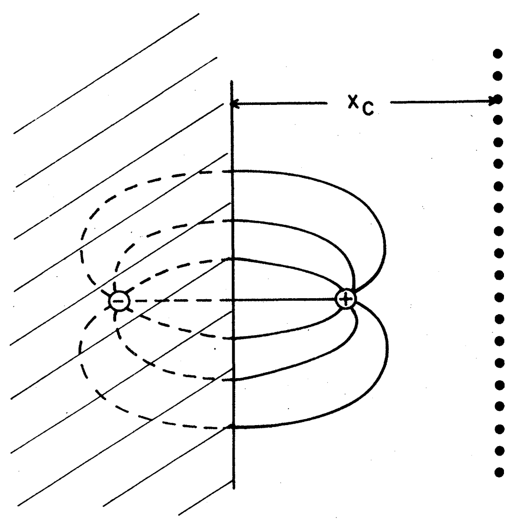


FIG. 2. Lines of constant phase for positively charged vortex trapped by its image charge near a wall occupying the shaded region. When an external uniform superflow is imposed parallel to the wall, the trapped vortex can escape over a dotted ridge line of potential maxima at a distance $x_c$ from the wall. (Distance $x_c$ is denoted $\frac{1}{2}r_c$ in the text.)

One is again faced with an "escape over the barrier" problem (Fig. 2), with the barrier now being a ridge running parallel to the wall with coordinates $(\frac{1}{2}r_c, y)$. Proceeding as in Sec. IV A,[27] one finds that the rate per unit wall length of escaping vortices is

$$R_w = D[y(l_c)/r_c^3][2K(l_c)]^{1/2} \exp[\pi K(l_c)] . \tag{4.13}$$

A factor of $\exp(\pi K)$ appears, rather than $\exp(2\pi K)$, because Eq. (4.11) is half the corresponding pair potential. The other factors occur because the escape is over a ridge, rather the across a saddle point.

To determine the instantaneous concentration of unbound vortices under conditions of uniform superflow, consider the rate of change of the free-vortex density. This may be written

$$\frac{dn_f}{dt} = R - v_\perp \sigma_c n_f^2 + \frac{R_w}{W} - \frac{v_\perp n_f}{W} , \tag{4.14}$$

where $W$ is the width of the superfluid strip, $v_\perp$ is the average velocity of vortices perpendicular to the superflow, and $\sigma_c$ is the cross section for vortex-pair recombination. The first and third terms are just the bulk and wall generation rates discussed above. The final term is the flux of vortices into the wall; $v_\perp$ can be estimated from Eq. (2.3).

$$v_\perp = \left| \left\langle \frac{d\vec{r}}{dt} \right\rangle_\perp \right| \approx \frac{2\pi D \hbar \rho_s^0}{m k_B T} |\vec{v}_n - \vec{u}_s| \propto 2\pi K(l_c) D r_c^{-1} , \tag{4.15}$$

where we have used the definition of $r_c$ given below Eq. (4.8), and the fact that $(\hbar/m)^2[\rho_s^0/k_B T\tilde{\epsilon}(r_c)] = K(l_c)$. The term in Eq. (4.14) proportional to $n_f^2$ is due to recombination of free vortices; the associated capture cross section $\sigma_c$ is clearly of order $r_c$. Our equations describing superflow in the nonlinear regime at low frequencies are completed by Eq. (2.10), which in this limit, reads

$$\frac{du_s}{dt} \approx \frac{2\pi\hbar}{m} n_f v_\perp . \tag{4.16}$$

Note that $n_f$, $v_\perp$, $\sigma_c$, $R$, and $r_w$ are all time dependent, through their dependence on $|\vec{v}_n(t) - \vec{u}_s(t)|$.

In the limit of large $W$, the wall terms in Eq. (4.14) are negligible relative to the bulk contributions. Under conditions of large amplitude, low-frequency driving, it is easy to check that after an initial transient,

$$\left| \frac{1}{n_f} \frac{dn_f}{dt} \right| >> \frac{1}{u_s} \frac{du_s}{dt} . \tag{4.17}$$

Thus it is reasonable to assume that Eq. (4.14) relaxes very rapidly to a quasisteady state with $n_f$ determined by the instantaneous value of $u_s(t)$ and $v_n(t)$;

$$n_f(t) \approx (R/v_\perp \sigma_c)^{1/2} \approx [R/2\pi K(l_c)]^{1/2} , \tag{4.18}$$

where we have used Eq. (4.15) and set $\sigma_c \approx r_c$. In the regime of Eq. (4.8), this becomes

$$n_f \sim \frac{x(T)}{a_0^2}\left(\frac{a_0}{r_c(t)}\right)^{2+(1/2)x(T)} \tag{4.19}$$

and remembering that $r_c \propto \hbar/m|\vec{v}_n - \vec{u}_s|$, we have[29]

$$\frac{du_s}{dt} \propto x(T)|\vec{v}_n - \vec{u}_s|^{3+(1/2)x(T)} \ , \tag{4.20}$$

in agreement with Eq. (12) of I.

Equations (4.14) and (4.15) can also be used to study the relaxation of an initial state of nonzero $\vec{u}_s$ with $v_n(t) \equiv 0$. Then, the relaxation rates $(1/n_f)\, dn_f/dt$ and $(1/u_s)\, du_s/dt$ are comparable and one must deal with two coupled nonlinear equations. Equation (4.20) with $v_n = 0$ should still give the correct qualitative behavior, however.

Thus far we have neglected the wall contributions to Eq. (4.14). For sufficiently narrow samples ($W \lesssim 1/\sigma_c n_f$), annihilation of free vortices may occur predominantly at the walls rather than through pair recombination. In this case, however, the wall generation term $R_w/W$ dominates pair dissociation. In both the driven problem and freely decaying superflow, $n_f$ now relaxes rapidly to a quasiequilibrium value

$$n_f(t) \approx R_w/v_\perp \ . \tag{4.21}$$

As one can easily check from Eq. (4.13), this $n_f(t)$ has the same $r_c$ dependence as Eq. (4.19), so Eq. (4.20) again follows.

Comparing our results with the approach of Langer and Reppy,[3] we find from Eq. (4.18) that it is the *square root* of the rate considered by those authors which sets the scale for the decay of superflow. This result also holds when wall effects dominate, since the relevant saddle-point energy is half the corresponding bulk value. This observation may be of some relevance also in three dimensions. For example, in bulk, two vortex rings exceeding the critical size might combine to form one ring greater than the critical radius and another smaller than the critical size. This would be analogous to the pair-recombination process discussed above. When annihilation of vortex rings at the wall is important, then wall generation should also be included, as the barrier to ring formation is smaller at the walls than in the bulk.

## V. THIRD SOUND AND HYDRODYNAMICS ABOVE $T_c$

### A. Equations of motion

Thus far, we have studied the response of superfluid films to spatially homogeneous time-dependent perturbations. This is the problem of physical interest in the Andronikoshvili-type experiments of Bishop and Reppy[15] and of Webster *et al.*[30] Third-sound measurements,[31–33] on the other hand, probe the response of films to finite wave-vector excitations. In this section, we generalize the treatment of I in a way which allows us to track third-sound excitations through the Kosterlitz-Thouless transition, and study the hydrodynamic region immediately above $T_c$.

Following the original work of Atkins,[34] Bergman was able to study the linearized hydrodynamics of third sound in some detail.[35,36] In particular, both heat transfer to the substrate and vapor, as well as mass transfer to the vapor were taken into account. Here we show how the effects of free and bound vortices can be incorporated into the Bergman description.

Our aim is to obtain "semimicroscopic" equations for the local superfluid velocity $\vec{v}_s(\vec{r},t)$ and other quantities, with fluctuations on a scale less than some cutoff $\Lambda^{-1}$ filtered out. In particular, we shall choose $\Lambda^{-1}$ to be larger than the effective vortex core diameter after very short-wavelength fluctuations of rotons, etc. are integrated out. The quantity $\vec{v}_s(\vec{r},t)$ will be "semimicroscopic" in the sense that it can possess a nonzero curl from point to point due to the presence of vortices. Assuming a continuous condensate wave function $\psi(\vec{r},t)$ on scales greater than $\Lambda^{-1}$, we define $\vec{v}_s$ by Eq. (1.2)

$$\vec{v}_s(\vec{r},t) \equiv \frac{\hbar}{m}\frac{\mathrm{Im}\psi^*\vec{\nabla}\psi}{|\psi|^2} \ , \tag{5.1}$$

which is well defined everywhere except at the zeros of $|\psi|$. Since the line integral of $\vec{v}_s(\vec{r},t)$ around any closed path must be $2\pi$ times the enclosed vorticity, we must have

$$\vec{\nabla} \times \vec{v}_s(\vec{r},t) = N(\vec{r},t)\hat{z} \ , \tag{5.2}$$

where $N(\vec{r},t)$ it the vortex density,

$$N(\vec{r},t) = \frac{2\pi\hbar}{m}\sum_i n_i \delta[\vec{r} - \vec{r}_i(t)] \ . \tag{5.3}$$

Upon defining a vortex current density,

$$\vec{J}_v(\vec{r},t) = \frac{2\pi\hbar}{m}\sum_i n_i \frac{d\vec{r}_i(t)}{dt}\delta[\vec{r} - \vec{r}_i(t)] \ , \tag{5.4}$$

and noting that vortices are created or destroyed only in pairs or at the boundaries, we have in addition the equation of continuity

$$\frac{\partial N}{\partial t} = -\vec{\nabla}\cdot\vec{J}_v \ . \tag{5.5}$$

This equation, when averaged over a hydrodynamic volume containing many vortices, is related to the Fokker-Planck equation associated with Eq. (2.11).

Since Eqs. (5.2) and (5.5) can be combined to read

$$\vec{\nabla} \times \left(\frac{\partial \vec{v}_s}{\partial t} + \hat{z} \times \vec{J}_v\right) = 0 \ , \tag{5.6}$$

we can write

$$\frac{\partial \vec{v}_s}{\partial t} + \hat{z} \times \vec{J}_v = \vec{\nabla} \Xi \ , \qquad (5.7)$$

where $\Xi$ is a scalar function. To make contact with the Bergman work, we identify $\Xi(\vec{r},t)$ with the local chemical potential $\mu(\vec{r},t)$. (Of course, this identification follows from the Josephson equation if one is not too close to a vortex core.) But it is also the case that

$$\vec{\nabla} \mu = \bar{S} \vec{\nabla} T - f \vec{\nabla} h \ , \qquad (5.8)$$

where $\bar{S}$ is the partial entropy of the film per unit mass

$$\bar{S} = \frac{\partial S}{\partial M} \qquad (5.9)$$

($M$ is the total mass of a film with constant area), $T$ is the temperature, $h$ is the film thickness, and $f$ is the Van der Waals constant. Our final equation for $\partial \vec{v}_s / \partial t$ is thus

$$\frac{\partial \vec{v}_s}{\partial t} = \bar{S} \vec{\nabla} T - f \vec{\nabla} h - \hat{z} \times \vec{J}_v \ , \qquad (5.10)$$

which is a modification of Eq. (3) of I. Note that the conservation of $\vec{v}_s$ is explicitly broken by the current of vortices flowing perpendicular to it.

The Bergman equations for the film height and temperature[36] are unchanged by the presence of vortices, and may be written

$$\frac{\partial(\bar{\rho} h)}{\partial t} = -\rho_s^0 \vec{\nabla} \cdot \vec{v}_s - J_m \ , \qquad (5.11)$$

$$\bar{\rho} h C \frac{\partial T}{\partial t} = \rho_s^0 T \bar{S} \vec{\nabla} \cdot \vec{v}_s + \kappa h \nabla^2 T - L J_m - J_Q \ , \qquad (5.12)$$

where $\bar{\rho} h$ is the mass density per unit area in the film and $L$ is the latent heat of evaporation from the film per unit mass. The quantity $C$ is the film specific heat per unit mass, $\kappa$ is the thermal conductivity of the helium in the film, $J_m$ is the mass flow per unit area from the film to the vapor, and $J_Q$ is the sum of the heat currents from the film to the substrate and to the vapor. The normal-fluid velocity $\vec{v}_n$ is assumed clamped to zero by the viscous interaction with the substrate, and all gradients are, of course, two dimensional.

Equations (5.10)–(5.12) together with Eqs. (5.2), (5.5), and the vortex equations of motion derived in Sec. II, constitute our description of excitations in superfluid films, both above and below the transition temperature. Note that fluctuations in the *amplitude* of $\psi(\vec{r},t)$ have been neglected, enabling us to write the superfluid momentum $\vec{j}_s = \rho_s^0 \vec{v}_s$ at all temperatures, with $\rho_s^0 = |\psi|^2 = \text{const}$. Amplitude variations are, however, effectively taken into account to some degree by the vortex cores associated with the density $N(\vec{r},t)$.

In order to complete the equations for the film, one must specify the behavior of $J_m$ and $J_Q$. We shall focus our attention on a particularly simple "ideal" situation, where one can neglect mass transport through the vapor, and where thermal conduction through the substrate is sufficiently high so that there are no fluctuations in temperature. This means that we set $J_m = 0$ in Eq. (5.11) and $\vec{\nabla} T = 0$ in Eq. (5.10). Mass transport through the vapor should in fact be negligible for very thin films, where the transition temperature is low compared to the bulk transition temperature, and the vapor pressure is very small at the temperatures of interest. Temperature fluctuations associated with third sound may be neglected at low temperatures even without good thermal contact to the substrate, because there is very little coupling between height and temperature fluctuations. Extension of our analysis to the more general situations considered by Bergman should be relatively straightforward.

### B. Analogy with the Maxwell equations and hydrodynamics above $T_c$

Equations (5.10), (5.11), and (5.12) simplify considerably when temperature fluctuations and mass flow into the vapor are neglected. Let us introduce the variable.

$$m(\vec{r},t) = [h_0 - h(\vec{r},t)] f/g \ , \qquad (5.13)$$

where $h_0$ is the average film thickness, and

$$g^2 = f \Bigg/ \left[ \bar{\rho} + d\left( \frac{d\bar{\rho}}{dh} \right) \right] \Bigg|_{h=h_0} \ . \qquad (5.14)$$

Equations (5.10) and (5.11) may now be written

$$\frac{\partial \vec{v}_s}{\partial t} = g \vec{\nabla} m - \hat{z} \times \vec{J}_v \ , \qquad (5.15)$$

$$\frac{\partial m}{\partial t} = \rho_s^0 g \vec{\nabla} \cdot \vec{v}_s \ . \qquad (5.16)$$

These equations, together with Eqs. (5.2) and (5.5), bear a striking resemblance to the Maxwell equations for two-dimensional electrodynamics (see Table I). Identifying $\hat{z} \times \vec{v}_s$ with the electric field and $m(\vec{r},t)$ with the $\hat{z}$ component of the magnetic field, we see that Eq. (5.2) is Coulomb's law, Eq. (5.15) is Ampere's law with a "displacement current," and Eq. (5.16) is Faraday's law. The zero-divergence condition on the magnetic field here amounts to the trivial assertion that

$$\left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (0, 0, m(x,y)) = 0 \ , \qquad (5.17)$$

while Eq. (5.5) is, of course, the equation of charge continuity.

TABLE I. Maxwell equations—third-sound analogy.

| Maxwell equation | Third-sound equation |
| --- | --- |
| $\vec{\nabla}\cdot\vec{E}=4\pi\rho$ | $\vec{\nabla}\times\vec{v}_s=N\hat{z}$ |
| $\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$ | $g\vec{\nabla}m=\hat{z}\times\vec{J}_v+\frac{\partial\vec{v}_s}{\partial t}$ |
| $\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0$ | $\rho_s^0 g\vec{\nabla}\cdot\vec{v}_s-\frac{\partial m}{\partial t}=0$ |
| $\vec{\nabla}\cdot\vec{B}=0$ | $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(0,0,m(x,y))=0$ |
| $\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{J}=0$ | $\frac{\partial N}{\partial t}+\vec{\nabla}\cdot\vec{J}_v=0$ |

In I and previously in this paper, we were led by the geometry of the Andronikoshvili experiments to consider a "quasistatic" approximation to the equations quoted above. (The "quasistatic" version of the Maxwell equations analogy was discussed in Sec. II B.) In this case, $\vec{\nabla}\cdot\vec{v}_s=0$, and $\langle\vec{\nabla}\mu\rangle=0$ to a first approximation. The static solution (2.8) of "Gaussian law," which neglects retardation effects, is then appropriate. However, retardation must be included in more general situations, where $\vec{\nabla}\cdot\vec{v}_s\neq 0$.

Let us first consider the equations of motion applicable in the limit of long wavelengths and low frequencies, i.e., the hydrodynamic region, for a fixed temperature slightly above $T_c$. In this region, the physics will be dominated by the dilute gas of free vortices, with the density $n_f\propto\xi_+^{-2}$. The effects of bound vortices are described by the dielectric function $\epsilon_b$, which in this region can be approximated by a constant, $\epsilon_b\approx\epsilon_c\equiv\rho_s^0/\rho_s(T_c^-)$. The "Maxwell equations" (5.15) and (5.16) etc., can then be modified in the usual way[37] to account for effects of the bound-vortex charge. The resulting equations read

$$\epsilon_c\frac{d\vec{v}_s}{dt}=g\vec{\nabla}m-\hat{z}\times\vec{J}_{v,\text{free}}\ , \tag{5.18}$$

$$\frac{\partial m}{\partial t}=\rho_s^0 g\vec{\nabla}\cdot\vec{v}_s\ , \tag{5.19}$$

$$\epsilon_c\vec{\nabla}\times\vec{v}_s=N_{\text{free}}\hat{z}\ , \tag{5.20}$$

where $N_{\text{free}}$ and $J_{v,\text{free}}$ are a density and current of free vortices which satisfy

$$\frac{\partial N_{\text{free}}}{\partial t}=\vec{\nabla}\cdot\vec{J}_{v,\text{free}}\ . \tag{5.21}$$

To obtain a closed system of equations for $\vec{v}_s$ and $m$, it is necessary to determine $\vec{J}_v$ in terms of $\vec{v}_s$. We start with the Langevin equation (2.4) specialized to the case $\vec{v}_n=0$ with $C=1$ for simplicity

$$\frac{d\vec{r}_i}{dt}=-n_i\frac{2\pi D\hbar\rho_s^0}{mk_BT}\hat{z}\times\vec{v}_s+\vec{\eta}_i(t)\ . \tag{5.22}$$

Upon setting

$$N_{\text{free}}(\vec{r},t)=N_+(\vec{r},t)-N_-(\vec{r},t)\equiv\delta N\ , \tag{5.23}$$

where $N_+$ and $N_-$ are densities of free positive and negative vortices, respectively, one readily constructs a Fokker-Planck equation linearized in $\delta N$ and $\vec{v}_s$. The resulting equation is

$$\frac{\partial\delta N}{\partial t}=\gamma_0\vec{\nabla}\cdot(\hat{z}\times\vec{v}_s)+D\vec{\nabla}^2\delta N\ , \tag{5.24}$$

where

$$\gamma_0\equiv\frac{D\rho_s^0 n_f}{k_BT}\left(\frac{2\pi\hbar}{m}\right)^2\simeq\xi_+^{-2}\ , \tag{5.25}$$

and one now readily identifies a current density

$$\vec{J}_{v,\text{free}}=-\gamma_0\hat{z}\times\vec{v}_s-D\vec{\nabla}\delta N\ . \tag{5.26}$$

Passing to Fourier transformed variables $\delta\hat{N}(\vec{k},\omega)$ and $\vec{v}_s(\vec{k},\omega)$, it follows from Eq. (5.24) that:

$$\hat{z}\delta\hat{N}(\vec{k},\omega)=\frac{-\gamma_0 i\vec{k}\times\vec{v}_s(\vec{k},\omega)}{-i\omega+Dk^2}\ . \tag{5.27}$$

The corresponding current density is

$$\hat{J}^i_{v,\text{free}}(\vec{k},\omega)=\gamma_0\left(\epsilon_{ij}-\frac{Dk_ik_l\epsilon_{lj}}{-i\omega+Dk^2}\right)\hat{v}_s^j(\vec{k},\omega)\ , \tag{5.28}$$

where $\epsilon_{ij}$ is the $2\times 2$ antisymmetric unit matrix, $\epsilon_{12}=-\epsilon_{21}=1$. The coefficient of $\hat{v}_s^j(k,\omega)$ is related to the conductivity in the Maxwell equations analogue.

Inserting this result in Eq. (5.18), we find a closed set of equations for $\vec{v}_s(\vec{k},\omega)$ and $m(\vec{k},\omega)$. These can be simplified further by introducing longitudinal and transverse projections of the superfluid velocity, namely

$$\hat{v}_L=\frac{\vec{k}\cdot\vec{v}_s}{k},\quad \hat{v}_T=\frac{\vec{k}\times\vec{v}_s}{k}\ . \tag{5.29}$$

Only the longitudinal part couples to $m(\vec{k},\omega)$, producing eigenfrequencies $\omega_\pm(k,\xi_+)$ which are the solutions of

$$\epsilon_c\omega_\pm^2+i\gamma_0\omega_\pm-g^2\rho_s^0k^2=0\ . \tag{5.30}$$

For small $k$, one finds

$$\omega_+(k,\xi_+)\approx\frac{-\gamma_0}{\epsilon_c}i\propto-\xi_+^{-2}i\ , \tag{5.31a}$$

$$\omega_-(k,\xi_+)\approx-iD_hk^2\ , \tag{5.31b}$$

$$D_h\equiv\rho_s^0g^2/\gamma_0\propto\xi_+^2\ . \tag{5.31c}$$

The transverse part of $\vec{v}_s$ relaxes at a rate

$$\omega_0(k,\xi_+)=-\left(\frac{\gamma_0}{\epsilon_c}+Dk^2\right)i\ , \tag{5.31d}$$

$$\approx-\frac{\gamma_0}{\epsilon_c}i\propto-\xi_+^{-2}i\ . \tag{5.31e}$$

Making use of Eq. (5.20) in Eq. (5.27), we see that this is also the relaxation rate of fluctuations in $N_{\text{free}}$.

The relaxational modes $\omega_+$ and $\omega_0$ exhibit critical slowing down as $T \rightarrow T_c^+$, with the same temperature dependence as $\xi_+^{-2}$. The conserved quantity $m(\vec{r},t)$, which is proportional to the local film thickness, only appears in the mode $\omega_-$, in the hydrodynamic region for $T \rightarrow T_c$. The behavior is diffusive, with a diffusion constant $D_h$ which diverges as $\xi_+^2$.

It is useful to define a mass-transport coefficient $\lambda$ by

$$\frac{\partial(\bar{\rho}h)}{\partial t} = -\lambda \vec{\nabla}^2 \mu = \lambda f \vec{\nabla}^2 h \quad . \tag{5.32a}$$

Comparing the relaxation rate obtained from Eq. (5.32a) with Eq. (5.31b), we find that

$$\lambda = D_h / g^2 \quad , \tag{5.32b}$$

so that $\lambda$ also diverges as $\xi_+^2$, for $T \rightarrow T_c^+$.

The transport coefficient $\lambda$ can be measured in principle by a heat-transport experiment, if the vapor pressure is not too small. Here, a temperature gradient is used to drive a mass current $j_s = \lambda \vec{\nabla} \mu$ in the film. Under suitable conditions, one may assume that the film thickness varies in such a way that the vapor pressure above the film is constant. One may then show that to a good approximation

$$\vec{\nabla} \mu = -(L/T) \vec{\nabla} T \quad , \tag{5.33}$$

where $L$ is the latent heat. In the steady state, the mass flow in the film is balanced by a counterflow in the gas, which is maintained in turn by evaporation at the hot end and condensation at the cold end of the film. Since each atom condensing deposits its latent heat in the film, the net heat transported is the same as if the film had a thermal conductivity $\kappa^{\text{eff}}$, given by

$$h\kappa^{\text{eff}} = \lambda L^2 / T \quad . \tag{5.34}$$

Experiments to measure thickness diffusion or thermal transport would provide both an important test of the dynamical theory, and an indirect measure of the crucial quantity $\xi_+(T)$.

### C. Third-sound propagation

We next consider the solutions to Eq. (5.30) for wave vectors such that $kg(\rho_s^0/\epsilon_0)^{1/2} >> \gamma_0$. (This is the situation for short wavelengths above $T_c$, and for all wavelengths at $T \leq T_c$, where $\gamma_0 = 0$.) We now find that Eq. (5.30) has two propagating (third-sound) solutions, with $\omega_\pm \approx \pm kg(\rho_s^0/\epsilon_c)^{1/2}$. A more accurate description of the velocity and damping of third sound in this region requires a more careful analysis of the effects of polarization of bound-vortex pairs. In principle, this means that the quantity $\epsilon_c$, in Eqs. (5.18), (5.20), and (5.30), etc., should be replaced by a wave-vector- and frequency-dependent bound-vortex dielectric function $\epsilon_b(\vec{k},\omega)$. At a frequency $\omega$ the dominant contribution to $\epsilon_b$ will come from bound pairs of separation $r \lesssim (14D/\omega)^{1/2}$; but $(D/\omega)^{1/2}$ is also the maximum distance that a vortex can diffuse in a quarter cycle of the third sound. The third-sound wavelength $\lambda_3 = 2\pi c_3/\omega$ will be very large compared to the distance $(D/\omega)^{1/2}$, provided that $\omega$ is small compared to the microscopic frequency $(\hbar/ma_0^2) \simeq 10^{11}\ \text{sec}^{-1}$, and we may therefore evaluate $\epsilon_b(k,\omega)$ at $k=0$. It is also clear that we may neglect retardation in the interaction between the two members of a vortex pair whose separation $r$ is of order $(14D/\omega)^{1/2}$, so that $\epsilon_b(0,\omega) \equiv \epsilon_b(\omega)$ is just the frequency-dependent dielectric function (3.15) that we estimated previously.

With this observation, Eqs. (5.18), (5.19), and (5.20) become, upon Fourier transformation

$$-i\omega\epsilon_b(\omega)\vec{v}_s = ig\vec{k}\hat{m} + \hat{z} \times \vec{J}_{v,\text{free}} \quad , \tag{5.35a}$$

$$-i\omega\hat{m} = i\rho_s^0 g \vec{k} \cdot \vec{v}_s \quad , \tag{5.35b}$$

$$-i\omega\epsilon_b(\omega)\vec{k} \times \vec{v}_s = \vec{N}_{\text{free}} \quad , \tag{5.35c}$$

where $\epsilon_b(\omega)$ is given by Eq. (3.15) with the understanding that $\xi_+ = \infty$ for $T \leq T_c$. Now, $\vec{J}_{v,\text{free}}$ can be computed just as before. Note from Eqs. (5.27) and (5.28) that $N_{\text{free}}$ and $\vec{J}_{v,\text{free}}$ are zero below $T_c$ in this linear theory, as they should be. Instead of Eq. (5.30), we now find eigenfrequencies $\omega_\pm(q,\xi_+)$ which satisfy

$$\epsilon_b(\omega_\pm)\omega_\pm^2 + i\gamma_0\omega_\pm - g^2\rho_s^0 k^2 = 0 \quad . \tag{5.36}$$

Below $T_c$, $\gamma_0$ vanishes, and the dispersion relations $\omega_\pm(k)$ are determined by inverting

$$k_\pm(\omega) = [\pm\epsilon_b(\omega)]^{1/2}\omega/c_3^0 \quad , \tag{5.37}$$

where $c_3^0 = g\sqrt{\rho_s^0}$. Decomposing $\epsilon_b(\omega)$ into its real and imaginary parts, and expanding in $\text{Im}\epsilon_b/\text{Re}\epsilon_b$ (which is small below $T_c$), we have

$$k_\pm(\omega) \approx \pm\frac{\omega}{c_3(\omega)} + \frac{1}{2}i\frac{\text{Im}\epsilon_b(\omega)}{c_3(\omega)\,\text{Re}\epsilon_b(\omega)}\omega \quad , \tag{5.38}$$

where

$$c_3(\omega) = c_3^0[\text{Re}\epsilon_b(\omega)]^{1/2} \quad . \tag{5.39}$$

As one would expect, $\text{Re}\epsilon_b(\omega)$ determines the renormalization of the third-sound velocity, while $\text{Im}\epsilon_b(\omega)$ controls the damping. As $k \rightarrow 0$ for fixed $T < T_c$, Eq. (5.37) can be inverted to read

$$\omega_\pm(k) = \pm c_3(0)k - D_3 k^{\pi K-1} i \quad , \tag{5.40}$$

where we have used Eqs. (3.13) and (3.14), $D_3$ is a coefficient obtainable from Eq. (5.38), and

$$K(T) = \hbar^2\rho_s(T)/m^2 k_B T \quad . \tag{5.41}$$

Of course, irreversible couplings neglected in Eqs.

(5.10), (5.11), and (5.12) will generate additional dissipation, proportional to $k^2$. Since $K$ drops from large values at low temperatures to $2/\pi$ at $T_c$, the damping displayed in Eq. (5.40) should dominate for, say $T \gtrsim \frac{1}{2} T_c$. Precisely at $T_c$, we find

$$\omega \approx c_3(0) k \left( \pm 1 - i \frac{\pi}{4 \ln^2(D/kc_3 a_0^2)} \right) . \quad (5.42)$$

We see that third sound should propagate fairly well for long wavelengths, at $T_c$.

For $T > T_c$, we find that third sound propagates fairly well, provided that the wave vector $k$ is larger than the quantity $\gamma_0/c_3 \propto \xi_+^{-2}$. For $k < \gamma_0/c_3$, there is no third-sound propagation, and the hydrodynamic results of Sec. V B apply.

The overall situation is illustrated in Fig. 3. Equation (5.36) provides a smooth interpolation (with an essential singularity at $T_c$) of the eigenfrequencies $\omega_\pm(k, \xi_+)$ through all regions of Fig. 3. In order to find the characteristic frequency $\omega_0(k, \xi_+)$ for transverse-velocity fluctuations, or for fluctuations in the vortex charge density $N$, in the region of large $k$, one must find the solution of the following equation, [which generalizes Eq. (5.32a)]

$$\omega_0 = -\left( \frac{\gamma_0}{\epsilon_b(k, \omega_0)} + Dk^2 \right) i \ . \quad (5.43)$$

In this case, it turns out that one cannot neglect the $k$ dependence of $\epsilon_b$. For example, if one ignores the first term on the right-hand side of Eq. (5.43),

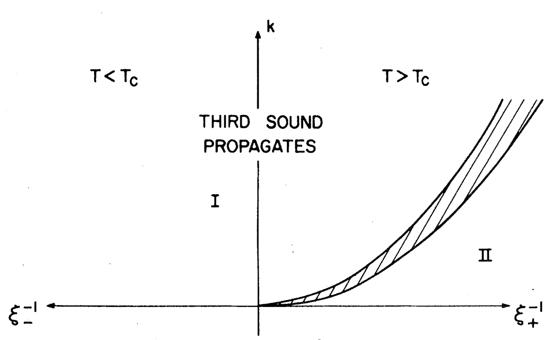


FIG. 3. Different regions of dynamic behavior at long wavelengths in the wave-vector-inverse-correlation-length plane. Region I, where third sound propagates, extends above $T_c$ to the blurred region where $k \approx (\text{const})\xi_+^{-2}$. In region II, there is diffusive transport of the film thickness, and a wave-vector-independent relaxation rate for the momentum density. In contrast to the behavior indicated here for third sound, the characteristic relaxation rate of transverse velocity fluctuations changes character when $k \approx \xi_+^{-1}$. The correlation length $\xi_-$ is just a convenient measure of temperature below $T_c$; there are only minor changes in the hydrodynamic behavior near $k\xi_- \approx 1$.

one finds a solution $\omega_0 = -iDk^2$. The important contribution to $\epsilon_b(\omega)$ in this frequency range, however, comes from pairs whose separation $r$ is of the same order as $k^{-1}$, in contrast to the situation discussed above for third sound.

In fact, for large $k$, the first term in Eq. (5.43) is probably more important than the second, since there are many more bound vortices than free vortices. Nevertheless, both terms suggest a characteristic relaxation rate of order $Dk^2$, and Eq. (5.32a) probably leads to a correct order-of-magnitude estimate of $\omega_0$ for any value of $k\xi_+$, provided that $k^{-1}$ and $\xi_+$ are large compared to the atomic spacing.

It is interesting to contrast our results with the predictions of dynamic scaling[38,39] in bulk helium. In bulk helium, second sound evolves into thermal diffusion and order-parameter relaxation modes as $T$ increases through $T_c$. In films, $\omega_-$ is analogous to the thermal mode, while $\omega_+$ and $\omega_0$ correspond roughly to relaxation of the phase and amplitude of the order parameter. Dynamic scaling breaks down badly in superfluid films: upon defining characterisitic frequencies $\omega_i^c$ by evaluating the $\omega_i(k, \xi_+)$ at $\xi_+ = k^{-1}$, we see that $\omega_+^c \sim \omega_0^c \sim k^2$, while $\omega_-^c \sim k^0$. If dynamic scaling held, the powers of $k$ would be identical. This point, and its connection with possible breakdowns of dynamic scaling in *three* dimensions, will be discussed further in another publication.[40]

## VI. SUMMARY AND CONCLUSIONS

We have studied the response of superfluid films to spatially homogeneous time-dependent perturbations, the nonlinear decay of superflow at low frequencies, and the dynamics of long-wavelength excitations such as third sound above and below $T_c$. Here, we summarize those features of the theory which could be tested experimentally, or might merit further theoretical investigation.

Superfluid film dynamics is described in greatest generality by the Eqs. (5.5), (5.10), (5.11), and (5.12) with heat and mass flow to the vapor and substrate included. With some simplifying assumptions (e.g., negligible mass flow to the vapor, and no temperature fluctuations), these equations take the form of the Maxwell equations in two dimensions. To these relations, one must add the Langevin equations of motion for vortex positions discussed in Sec. II. A semimicroscopic model of this kind leads to a Fokker-Planck equation for the charge density, and allows the vortex current to be eliminated in favor of the superfluid velocity field.

The analysis of weak, homogeneous, low-frequency perturbations in Secs. II and III may be viewed as a kind of "electrostatic" application of the theory. The approximations made there were dictated by the geometry of the Andronikashvili experiments of

Bishop and Reppy,[15] and may be summarized in terms of the frequency-dependent dielectric function $\epsilon(\omega)$ defined by Eq. (3.1). This quantity contains contributions from both free and bound vortices, and may be written

$$\epsilon(\omega)=\epsilon_b^+(\omega)+i\gamma_0/\omega \ , \tag{6.1}$$

or, equivalently, as in Eq. (3.17), which emphasizes the analogy to a charged plasma. The function $\epsilon_b^+(\omega)$ comes from bound pairs, and has the approximate spectral representation,

$$\epsilon_b^+(\omega)=1+\int_{a_0}^{\xi_+} dr\frac{d\tilde{\epsilon}(r)}{dr}\frac{14Dr^{-2}}{-i\omega+14Dr^{-2}} \ , \tag{6.2}$$

where $\tilde{\epsilon}(r)$ is defined in Appendix D and $\xi_+(T)$ is understood to be infinite below $T_c$. The quantity $\gamma_0$, which measures the rate at which supercurrents relax above $T_c$, is

$$\gamma_0=\left(\frac{2\pi\hbar}{m}\right)^2\left(\frac{D\rho_s^0}{k_BT}\right)n_f\simeq\xi_+^{-2} \ . \tag{6.3}$$

The average power dissipated in an Andronikoshvili experiment is then given by Eq. (3.4), and the period shifts by Eq. (3.6). A fit of these formulas to the data of Bishop and Reppy has been discussed by Ambegaokar and Teitel.[25]

It would be interesting to extend this analysis to the quartz microbalance experiments of Webster *et al.*,[30] who studied the universal jump prediction[8] in films of $^3$He-$^4$He mixtures. The generalization of our results to allow for a conserved $^3$He concentration appears straightforward. When applied to a simple model of mixture-film dynamics,[41] the analysis of Sec. V produces an additional diffusive mode both above and below $T_c$, without strong anomalies near the critical temperature.

The analysis of the nonlinear decay of superflow below $T_c$ (Sec. IV) overlaps with the work of McCauley,[42] and Myerson, Huberman *et al.*[11,29] It is perhaps most similar in spirit to the McCauley paper, which examines a two-dimensional Coulomb plasma in an electric field at low temperatures using Langevin equations similar to ours. Our principal result follows from combining the discussion of two-dimensional vortex nucleation by Langer and Reppy[3] with the Kosterlitz static-scaling theory.[5] At any fixed temperature below $T_c$, we find that the decay of a uniform supercurrent $u_s(t)$ may be expressed, for small $u_s$, as

$$\frac{du_s}{dt}\approx -A(T)u_s^{3+(1/2)x(T)} \ . \tag{6.4}$$

We have, for simplicity, set $\vec{v}_n\equiv 0$ in Eq. (4.20), and approximated the solution of the two coupled Eqs. (4.14) and (4.16) which must be solved in this case. The quantities $A(T)$ and $x(T)$ are positive, and depend on temperature, substrate, film thickness, etc. Although $A(T)$ can only be estimated, $x(T)$ is given exactly in terms of the areal superfluid density $\rho_s(T)$,

$$x(T)=-4+\frac{2\pi\hbar^2\rho_s(T)}{m^2k_BT}=-4+2\pi K(T) \ . \tag{6.5}$$

A consequence of the universal jump prediction is that $x(T)$ goes to zero with a square-root cusp[5] as $T\to T_c^-$,

$$x(T)\approx b(1-T/T_c)^{1/2} \ . \tag{6.6}$$

The quantity $A(T)$ behaves like $x(T)$ just below the critical temperature, and, precisely at $T_c$, there are logarithmic corrections to Eq. (6.4).[10]

It seems worth emphasizing, in passing, a universal amplitude ratio uncovered in I. If one describes the divergent correlation length $\xi_+(T)$ above $T_c$ by

$$\xi_+(T)\approx a\exp(b'/|1-T/T_c|^{1/2}) \ , \tag{6.7}$$

it is a universal consequence of the Kosterlitz recursion relations[5] that

$$b'=2\pi/b \ , \tag{6.8}$$

where $b$ is defined by Eq. (6.6). This universal relation has already been somewhat tested in the fit to experiment,[25] where a single parameter is adjusted to fit $\Delta P/P$ for $T<T_c$ and the shape of the high-temperature tails in $\Delta P/P$ and $Q^{-1}$ for $T>T_c$.

The hydrodynamic modes treated in Sec. V were not discussed in I. Perhaps the most significant new results concerns the "height" diffusion coefficient $D_h$ and the effective thermal transport coefficient $\kappa^{\text{eff}}$, above $T_c$, which are predicted to diverge as $\xi_+^2$ for $T\to T_c^+$. Experiments designed to detect this dramatic temperature dependence [see Eq. (6.7)] would provide an important test of the theory.

Our results on third-sound propagation below $T_c$, and for $k\gtrsim\xi_+^{-1}$ above $T_c$, may also be of some interest. The Bergman theory,[35,36] which includes thermal and mass currents to the substrate and vapor, seems unable to account for measurements of third-sound damping.[33] It would be interesting to see if the anomalous damping displayed in Eq. (5.40), when combined with the effects discussed by Bergman, could improve the agreement between theory and experiment.

*Note added in proof.* The diverging thermal coefficient reported by B. Ratnam and J. Mochel [Phys. Rev. Lett. 25, 711 (1970)] is not inconsistent with this prediction. [J. D. Reppy and F. Gasparini (private communications).]

## ACKNOWLEDGMENTS

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## APPENDIX A: DISORDERED SUBSTRATE

The static theory developed by Kosterlitz and Thouless[4,5] considered helium on a smooth substrate or equivalently planar spins[7] on a regular two-dimensional lattice. However, most experiments, so far, have been performed on disordered substrates such as Mylar. It is therefore important to ask whether substrate disorder will change the results of the Kosterlitz-Thouless theory, or of the dynamic theory of the present paper. Dash[43] has argued, in fact, that on a smooth substrate the superfluid transition in a very thin film will always be masked by a first-order transition between a dense liquidlike phase and a dilute two-dimensional vapor. However, it is clear that substrate irregularities will couple very strongly to density fluctuations in the helium adsorbate, and it seems plausible to us that frozen-in microscopic disorder may suppress the liquid-gas phase separation. It may actually be *necessary* to use a disordered substrate in order to see the Kosterlitz-Thouless transition in the thinnest films (i.e., for $T_c$ much less than the bulk value).

Let us recall the Harris argument[44] for the influence of a distribution of critical temperatures on a phase transition. The root-mean-square fluctuations $\overline{\delta T_c}$ in $T_c$ averaged over a block of size the correlation length scales like $\overline{\delta T_c} \sim \xi^{-d/2}$. The disorder should be irrelevant if $\overline{\delta T_c}$ is much less than the distance of the averaged $T_c$ from $T$, i.e., if $\xi^{-d/2}/(T-\overline{T}_c) << 1$. Since the correlation length is so strongly divergent near the Kosterlitz-Thouless transition, this criterion is evidently well satisfied. If a component of the disorder had a characteristic size $L_0$ much greater than an interparticle or lattice spacing $a$, then it would only begin to get averaged out after $\ln(L_0/a)$ iterations and the transition would appear broadened for $\xi \lesssim L_0/a$. A substrate such as Mylar with only short-ranged disorder might well appear more "ideal" than an assemblage of 200-Å microcrystals very close to $T_c$. Pinned vortices will be screened below $T_c$ and could in principle be integrated out along with the thermally excited pairs. Although they should not affect the thermodynamics they will be seen, in ways very hard to quantify, in a dynamic measurements.

A substrate, disordered or otherwise, will make $\rho_s/\rho$ less than unity even at zero temperature.[45] An explicit example may clarify this point.[46] Consider a *weakly interacting* Bose system on a periodic substrate. The energy and average momentum for a particle in a Bloch state with crystal momentum $k$ near the bottom of a parabolic band are[47]

$$\epsilon=\frac{\hbar^2k^2}{2m^*},\quad \langle\vec{p}\rangle=\frac{m\hbar\vec{k}}{m^*}\ , \tag{A1}$$

here $m^*$ is the effective mass and $m$ is the free-particle mass. Suppose that there are $n$ particles per unit area of substrate condensed in the state $\vec{k}$. Then the energy and momentum densities of the system are obtained from Eq. (A1) by multiplying by $n$. Comparing with the hydrodynamic forms (for $v_n=0$, and $v_s$ is small)

$$E\equiv\tfrac{1}{2}\rho_s v_s^2\ , \tag{A2a}$$

$$\vec{g}=\partial E/\partial\vec{v}_s=\rho_s\vec{v}_s\ , \tag{A2b}$$

where $E$ and $\vec{g}$ are the energy and momentum density, one finds

$$\rho_s=\frac{m^2}{m^*}n,\quad \vec{v}_s=\frac{\hbar\vec{k}}{m}\ . \tag{A3}$$

Note that the relation between $\vec{v}_s$ and the gradient of the phase of the order parameter involves the bare mass $m$.

We remark that on a fundamental level, the form of Eq. (A2b) is dictated by the principle of Galilean invariance; i.e., one requires that if $\vec{v}_s$ and $\vec{v}_n$ are both changed by an infinitesimal constant $d\vec{v}$, then $d\vec{g}=\rho\, d\vec{v}$ and $dE=\vec{g}\cdot d\vec{v}$. One also uses the fact that for any given $\vec{v}_s$, the energy must be a minimum at $v_n=0$. Equation (A2a) may be taken as the definition of $\rho_s$.

We would like to emphasize that the relation $\vec{v}_s=(\hbar/m)\vec{\nabla}\phi$ can also be proven more generally, using superfluid hydrodynamics and the Galilean transformation properties of the $N$-particle Schrödinger equation. Its validity is not restricted to periodic substrates or to weakly interacting bosons.

## APPENDIX B: DIFFUSION CONSTANT

The Fokker-Planck equation, satisfied by the distribution of vortices $\Gamma$, implied by Eq. (2.6) is

$$\frac{\partial\Gamma}{\partial t}+\sum_i \vec{\nabla}_i\cdot(\dot{\vec{r}}_i\Gamma-D\vec{\nabla}_i\Gamma)=0\ , \tag{B1}$$

where $\dot{\vec{r}}$ stands for the right-hand side of Eq. (2.3). We claim that when $v_n$ is time independent and we work in a frame where $v_n$ is zero, a static solution to Eq. (B1) is given by

$$\Gamma_0=\exp(-H/k_BT)\ , \tag{B2}$$

where

$$H=\frac{2\pi\rho_s^0\hbar}{m}\left[\sum_{i<j}\frac{\hbar}{m}n_in_jG(\vec{r}_i,\vec{r}_j)-\sum_i n_i(\hat{z}\times\vec{u}_s)\cdot\vec{r}_i\right]\ . \tag{B3}$$

It is readily verifed that

$$-D\vec{\nabla}_i\Gamma_0 = n_i D\frac{\rho_s^0 2\pi\hbar}{k_B T m}[\hat{z}\times\vec{v}_s(\vec{r}_i)]\Gamma_0 \ , \tag{B4}$$

where $\vec{v}_s(\vec{r}_i)$ is given by Eq. (2.8). Equation (B4) will cancel the term in Eq. (B1) coming from the first term of Eq. (2.3) for $\dot{\vec{r}}_i$. The second term in $\dot{\vec{r}}_i$ is just $(1-C)\vec{v}_s$ according to Eq. (2.3). It is readily shown from Eq. (2.8) that $\vec{\nabla}_i\cdot\vec{v}_s(\vec{r}_i)=0$. Thus Eq. (B1) reduces to

$$(1-C)\vec{v}_s(\vec{r}_i)\cdot\vec{\nabla}_i\Gamma_0 = 0 \ . \tag{B5}$$

It follows from (B4) that this last equality is satisfied.

To estimate $D$, note that for a film which is several atoms thick, the microscopic parameters available are $\hbar$, $m$, $a_0$, and $k_B T$. Bose condensation requires that the thermal de Broglie wavelength at $T_c$ be of the same order as the interparticle spacing. Thus $k_B T$ is not an independent quantity. From the remaining three microscopic quantities there is only one combination with the dimensions of (length$^2$ / time), namely $\hbar/m$.

A crude, though we believe correct, argument can now be given for why the diffusion constant $D$ remains finite at $T_c$ rather than diverging or vanishing, as is common in the dynamic critical behavior of various systems.[48]

Let us estimate the contribution to the diffusion of a test vortex, due to the velocity fields of the other vortices in the system. At any instant of time, the induced velocity $\vec{v}$ of the test vortex is proportional to the local superfluid velocity $\vec{v}_s$, according to Eq. (2.3). (We assume $\vec{v}_n=0$.) Thus we may write

$$\vec{v}(t)\propto\int d^2r\int_{R>2r} d^2R\ \Gamma(\vec{R},\vec{r};t)(\vec{r}\cdot\vec{\nabla}_R)\left[\frac{\vec{R}}{R^2}\right] \ , \tag{B6}$$

where $\Gamma(\vec{R},\vec{r};t)$ is the density of pairs with separation $\vec{r}$ and center of gravity at point $\vec{R}$, relative to the test vortex as origin. We have restricted the integral in (B6) to $R>2r$, so that we may make the dipole approximation for the velocity field of the pair. (The contribution from pairs with $R<2r$ must be computed separately.)

The contribution of Eq. (B6) to the diffusion constant may be written

$$\delta D = \frac{1}{4}\int_{-\infty}^{\infty}\langle\vec{v}(t)\cdot\vec{v}(0)\rangle\, dt \ , \tag{B7}$$

which may be related in turn to the correlation function $\langle\Gamma(\vec{R},\vec{r};t)\Gamma(\vec{R}',\vec{r}';0)\rangle$. We shall estimate this correlation function by ignoring the effects of the test vortex on the pair, and by ignoring the motion of the center of gravity $\vec{R}$. This leads to the approximation,

$$\int\langle\Gamma(\vec{r},\vec{R},t)\Gamma(r',R',0)\rangle r_i r_j'\, d^2r' \approx \tfrac{1}{2}\delta_{ij}\delta(\vec{R}-\vec{R}')\Gamma_0(\vec{r})r^2 e^{-14tD/r^2} \ , \tag{B8}$$

where $\Gamma_0(\vec{r})$ is the equilibrium distribution of vortex pairs, and we have used the results of Ref. 22 for the time dependence. With this approximation, one obtains the final result

$$\delta D\propto\int_{a_0}^{\infty}\Gamma_0(\vec{r})r^2\, d^2r \ . \tag{B9}$$

Since $\Gamma_0(r)$ falls off as $(r^4\ln^2 r)^{-1}$, at $T_c$ this integral converges.

The contribution of vortex pairs with $R<2r$ leads to a contribution to $D$ of the same form as Eq. (B9). Center-of-mass motion, and the polarization of the dipole gas by the test vortex will tend to *reduce* $\delta D$ by an amount comparable in magnitude to Eq. (B9). The net result, therefore, is that $\delta D$ is finite at $T_c$, and $\delta D$ will, moreover, be small if the vortex fugacity $y_0$ is small.

The authors are grateful to Annette Zippelius and Rolfe Petschek for very helpful discussions on these points. Dr. Zippelius, in particular, was the first to argue that $\delta D$ is determined by an integral of the form (B9).

## APPENDIX C: SUPERFLUID TRANSITION IN FILMS OF FINITE THICKNESS

The Kosterlitz-Thouless transition is often described as a vortex unbinding which occurs with increasing temperatures in strictly two-dimensional systems. In experimental situations, however, films are of finite thickness, and it is often possible to vary the film height at fixed temperature. One then expects a dissociation of vortices with decreasing thickness $h$. In this appendix, we argue that the universal jump prediction[8] and other consequences of the Kosterlitz-Thouless theory should hold for films of *arbitrary* thickness, provided one is sufficiently close to the transition temperature. Of course, the lateral dimensions of the film must always be large compared to $h$. Finite-size scaling theory[49] then describes the crossover from bulk behavior for large $h$, in terms of a universal crossover scaling function.

A thick film of superfluid $^4$He should behave like an infinite system, provided that the bulk superfluid coherence length is much less than $h$. This coherence length diverges in bulk as $T$ approaches $T_\lambda$, and will eventually reach the film thickness in finite samples. At this point, the Kosterlitz-Thouless analysis becomes applicable. In terms of the bulk transverse correlation length $\xi_\perp(T)$ used by Hohenberg *et al.*,[50] one requires that

$$\xi_\perp(T) = \frac{m^2 k_B T}{\hbar^2\rho_s^{(b)}(T)}\gtrsim h \tag{C1}$$

in order for the film to be effectively two dimensional. Here $\rho_s^{(b)}(T)$ is the *bulk* superfluid density measured in units of $\mathrm{gm/cm^3}$. Near $T_\lambda$, we may write[51]

$$\xi_\perp(T) = \xi_\perp^0[(T_\lambda - T)/T_\lambda]^{-\nu} \quad , \tag{C2}$$

where $\nu \approx \frac{2}{3}$ is the correlation-length exponent of a three-dimensional superfluid, and $\xi_\perp^0 \approx 3.57$ Å for pure $^4$He at its vapor pressure.[50]

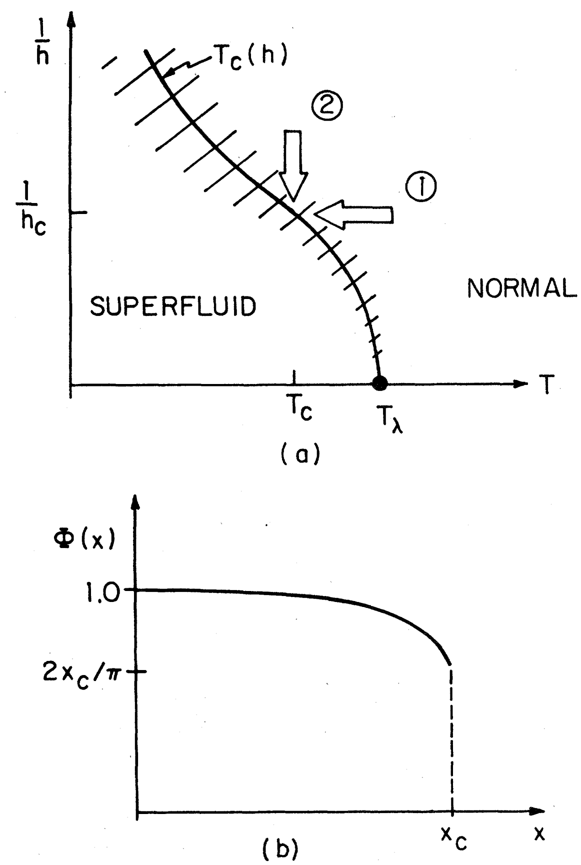


FIG. 4. (a) Phase diagram for a superfluid slab or "film" of thickness $h$ in the $z$ direction, and infinite in the remaining two directions. Long-range superfluid order exists only on a line at $h^{-1}=0$, below the bulk $\lambda$ temperature $T_\lambda$. The remainder of the region labeled "superfluid" is characterized by algebraic decay of the superfluid order parameter. Experimentally, one can reach the normal region either by decreasing the film thickness or increasing the temperature to cross the line $T_c(h)$ of Kosterlitz-Thouless transitions. Two possible paths of approach to the transition point $(T_c, h_c^{-1})$ from the normal phase are shown by the arrows 1 and 2. The shaded region is the region where two-dimensional fluctuations predominate, and the Kosterlitz-Thouless theory is applicable. (b) Sketch of universal scaling function $\Phi(x)$ describing crossover from three- to two-dimensional behavior in the superfluid density. This function is equal to unity at $x=0$, and exhibits a jump discontinuity at a point $x_c$, from a value $2x_c/\pi$ to the value 0. The jump is preceded by a square-root cusp.

The phase diagram in the temperature-reciprocal-film-thickness plane is shown in Fig. 4(a). The bulk $\lambda$ transition is an isolated point at $h=\infty$ connected to a line of Kosterlitz-Thouless transitions. The reasoning sketched above would lead one to expect a universal-jump discontinuity in the areal superfluid density $\rho_s(T,h)$ for *all* finite film thicknesses.

Two possible experimental paths of approach to a point $(T_c, h_c^{-1})$ on the critical line are shown in Fig. 4(a). It is easy to convert theoretical predictions made for path 1 into the corresponding result for path 2 by invoking a universality or "smoothness" hypothesis.[52] For example, if it is known that the two-dimensional correlation length

$$\xi_+(T,h) \approx a \exp(2\pi/bt^{1/2}) \tag{C3}$$

on path 1, where $t=[T-T_c(h)]/T_c(h)$, it follows that:

$$\xi_+(T,h) \approx a \exp(2\pi/b\Delta^{1/2}) \quad , \tag{C4}$$

where

$$\Delta = (h_c - h)\frac{1}{T_c}\frac{dT_c(h)}{dh}\bigg|_{h=h_c} \tag{C5}$$

on path 2.

The crossover from bulk to Kosterlitz-Thouless behavior should be describable by a universal scaling function for large $h$. Indeed, a straightforward application of finite-size-scaling theory[49] suggests that for thick films, the areal superfluid density has the form

$$\rho_s(T,h) = h\rho_s^{(b)}(T)\Phi[\xi_\perp(T)/h] \quad , \tag{C6}$$

where $\Phi(x)$ is a universal crossover function. Clearly, $\Phi(x) \to 1$, for $x \to 0$, and Eq. (C6) is consistent with the condition (C1) for deviations from bulk behavior, provided that $\Phi(x)$ begins to deviate significantly from unity when its argument is of order unity.

In order for $\Phi(x)$ to describe the Kosterlitz-Thouless transition, it must jump discontinuously to zero at some finite value $x_c$ of its argument. Furthermore, Eq. (C1) implies the universal value[8] of the jump and

$$\lim_{x \to x_c^-} \Phi(x) = 2x_c/\pi \quad . \tag{C7}$$

Of course, $\Phi(x)$ should have a square-root cusp[5] as $x \to x_c^-$. The form of $\Phi(x)$ is sketched in Fig. 4(b).

One consequence of the crossover scaling analysis is that for large $h$,[48]

$$T_c(h) \approx T_\lambda(1 - A/h^{1/\nu}) \approx T_\lambda(1 - Ah^{-1.5}) \quad , \tag{C8}$$

where $A = (\xi_\perp^0/x_c)^{1/\nu}$.

## APPENDIX D: KOSTERLITZ-THOULESS THEORY

Here we collect a few results from Refs. 5 and 9 which have been used repeatedly in the main text. In our work, we have relied heavily on analogies with macroscopic electrodynamics. In the static limit, this is the point of view taken in the original paper of Kosterlitz and Thouless.[4] As Young[9] has emphasized, the "iterated mean field" or dielectric approximation of Ref. 4 is precisely equivalent to the real-space renormalization-group equations of Ref. 5 if an unnecessary simplifying assumption made in Ref. 4 is avoided.

The scale-dependent dielectric constant $\tilde{\epsilon}(r)$ introduced by Kosterlitz and Thouless is determined self-consistently in terms of three microscopic parameters: $K_0 \equiv (\hbar/m)^2(\rho_s^0/k_B T)$, the core diameter $a_0$, and $y_0$, related to the vortex core energy $C_0$ by $y_0 = \exp(-C_0/k_B T)$ or to an equivalent chemical potential by $y_0 = \exp(\mu_0/k_B T)$. The effect of pairs on length scales smaller than $a_0 e^l$ can be absorbed into new parameters $y(l)$ and $K(l)$ which obey the Kosterlitz recursion relations.

$$\frac{d}{dl}[K(l)]^{-1} = 4\pi^3 y^2(l) \ , \quad \text{(D1)}$$

$$\frac{d}{dl} y(l) = [2 - \pi K(l)] y(l) \ , \quad \text{(D2)}$$

where $K(l=0) = K_0$ and $y(l=0) = y_0$. In fact the integrated form of Eqs. (D1) and (D2) is precisely the mean-field approximation of Kosterlitz and Thouless[4,9]

$$[K(l)]^{-1} = K_0^{-1} + 4\pi^3 \int_0^l dl' \, y^2(l') \ , \quad \text{(D3)}$$

$$y(l) = y_0 \exp\left[2l - \pi \int_0^l dl' \, K(l')\right] \ . \quad \text{(D4)}$$

From Eqs. (D3) and (D4) one sees how small pairs can be absorbed into the scale-dependent coupling constants. These equations show that

$$\begin{aligned} K(l, K_0, y_0, a_0) &= K(l, K(l'), y(l'), a_0 e^{l'}) \ , \\ y(l, K_0, y_0, a_0) &= y(l, K(l'), y(l'), a_0 e^{l'}) \ , \end{aligned} \quad \text{(D5)}$$

for any $l' < l$, so that by rescaling $K_0$, $y_0$, and $a_0$ we may absorb the screening effect of pairs of separation $\leq a_0 e^{l'}$ into the new coupling constants $K(l')$ and $y(l')$.

The scale-dependent dielectric constant is defined by

$$\tilde{\epsilon}(r) = K_0 / K(l = \ln(r/a_0)) \ . \quad \text{(D6)}$$

The vortex-depairing transition temperature $T_c$ is determined by $K(\infty) = (2/\pi)$.

Near the transition we write $K(l) = (2/\pi)[1 + \frac{1}{2}x(l)]$. Then the Eqs. (D1) and (D2) are solved by

$$\begin{aligned} x(l) &= \tfrac{1}{2}x(T) \coth\{\tfrac{1}{2}x(T) l + \coth^{-1}[2x_0/x(T)]\}, \\ 4\pi y(l) &= \tfrac{1}{2}x(T) \operatorname{csch}\{\tfrac{1}{2}x(T) l + \coth^{-1}[2x_0/x(T)]\}, \\ & \qquad t < 0 \ , \end{aligned} \quad \text{(D7)}$$

$$\begin{aligned} x(l) &= \tfrac{1}{2}x(T) \cot\{\tfrac{1}{2}x(T) l + \cot^{-1}[2x_0/x(T)]\}, \\ 4\pi y(l) &= \tfrac{1}{2}x(T) \csc\{\tfrac{1}{2}x(T) l + \cot^{-1}[2x_0/x(T)]\}, \\ & \qquad t > 0 \ , \end{aligned} \quad \text{(D8)}$$

where $t \equiv (T - T_c)/T_c$ and $x(T) = b(|t|)^{1/2}$. Below $T_c$, $\tilde{\epsilon}(\infty)$ is finite and the renormalized superfluid density is given by $\rho_s(T) = \rho_s^0/\tilde{\epsilon}(\infty)$. Using Eq. (D7) and definitions given above, one finds

$$\rho_s(T) = \rho_s(T_c^-)[1 + \tfrac{1}{4}x(T)] \ , \quad \text{(D9)}$$

where $\rho_s(T_c^-)$ refers to the value at the transition, namely $(m/\hbar)^2(k_B T)(2/\pi)$.

The correlation length $\xi_-$ is defined[10] by writing for $t < 0$

$$\tilde{\epsilon}(r) = \tilde{\epsilon}(\infty)\{1 - \tfrac{1}{2}x(T) \exp[-\ln(r/a_0)/\ln(\xi_-/a_0)]\} \ , \quad \text{(D10)}$$

from which it follows, using Eq. (D7), that:

$$\xi_- \approx a_0 \exp[1/x(T)] \ . \quad \text{(D11)}$$

Above $T_c$, Kosterlitz[5] has defined a correlation length $\xi_+$ as the scale at which $y(l)$ becomes comparable to $y_0$. From Eq. (D8) we find that

$$\xi_+ \approx a_0 \exp[2\pi/x(T)] \ . \quad \text{(D12)}$$


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[19]This point of view is implicit in the work of A. M. Polyakov, Phys. Lett. B 57, 79 (1975) on fixed-length spin systems. R. A. Pelcovits [Ph.D. thesis, (Harvard University, 1977) (unpublished)] has shown explicitly that amplitude fluctuations are irrelevant everywhere along the line of fixed points which characterizes the low-temperature phase of two-dimensional superfluids.

[20]See W. F. Vinen, Prog. Low Temp. Phys. 3, 1 (1961), and references therein.

[21]To understand the units of the equations in this section one must note that two-dimensional charges (e.g., $q_0$) have units $e(L)^{-1/2}$ where $e$ is a three-dimensional unit of charge and $L$ is a unit of length. Correspondingly, the electric field in this section has units of $(L)^{1/2}$ multiplied by a three-dimensional electric field.

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[23]Conventionally, since the dynamics is diffusive, this is called a Smoluchowski equation. See, e.g., S. Chandrasekhar, Rev. Mod. Phys. 15, 58 (1943).

[24]It can be shown that the vortex charge density may also be written as

$$N(\vec{r}) = \delta[\psi_1(\vec{r})]\delta[\psi_2(\vec{r})]\det\partial_\alpha\psi_\beta(\vec{r}) \quad ,$$

where $\psi_1$ and $\psi_2$ are the real and imaginary parts of $\psi$, and $\delta$ is the Dirac $\delta$ function. From this, one can argue that the large-$r$ behavior of the correlation function $\langle N(\vec{r})N(0)\rangle$ should be dominated by the decay of the four-point expectation value

$$\langle \partial_\alpha\psi_1(\vec{r})\partial_\beta\psi_2(\vec{r})\partial_\gamma\psi_1(0)\partial_\delta\psi_2(0)\rangle \quad .$$

Ordinarily one would expect such a function to decay with an exponential factor equal to the square of the two-point function $\langle\psi^*(\vec{r})\psi(0)\rangle$. In the language of field theory, we say that the spectrum of the two-particle propagator begins with a cut at twice the mass of the one-particle propagator, provided there are no "two-particle bound states," (or at least none with the correct symmetry to couple to $\det\partial_\alpha\psi_\beta$). Since the basic interaction between $\psi_1$ and $\psi_2$ is repulsive, it seems likely that there are no bound states in the present case.

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[29]Since the works of Ref. 11 are concerned with calculating the same decay rates that we are describing in this section, and have previously published in I, it may be worth pointing out that the results are different. Thus. Eq. (8) of Huberman *et al.* should be identical, apart from differences in notation, to our Eq. (4.8) but is missing the factor $[x(T)]^2$. Further, Myerson's Eq. (1.4) should be identical to our Eq. (4.20) but has an exponent $5+x(T)$ in the numerator. The transcription of notations is $x(T) \leftrightarrow 2(\beta-2)$, where $\beta$ is the parameter defined in Eqs. (5) and (6) of Huberman *et al.* and called $\beta^*$ by Myerson.

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