Pseudospin formulation of kinetic Ising models

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It is shown that Glauber's kinetic Ising model is equivalent to an Ising model with multispin interactions in a transverse field. In one dimension, certain of Glauber's results are recovered by using the well-known fermion representation for spin-$\frac{1}{2}$.

Among the simplest models that exhibit nontrivial critical dynamics, is the kinetic Ising model proposed by Glauber and solved by him in one dimension on a lattice.[1] In this model, the probability distribution for N Ising spins relaxes by means of spontaneous spin flips whose probability depends on the configuration of neighboring spins. It is easy to show, and implicit in the work of Kadanoff and Swift, that in any dimension Glauber's model is equivalent to an Ising model with competing interactions in a transverse field.[2] The probability distribution for Glauber's variables is related to a quantum many-body wave function and the corresponding energy eigenvalues become the relaxation rates. The long-time behavior of correlation functions is governed by the low-lying excited states of the corresponding quantum problem whose ground state is the equilibrium distribution. This restatement of the critical-dynamics problem retains the simplicity of a two-state variable but has only proven useful computationally in one dimension where a transformation to fermions permits one to recover some of Glauber's results.[3] A second model that conserves energy in the relaxation process is also discussed.

The probability that a system of N spins, σ_i, is in one of its 2^N possible configurations at time t is denoted by $P(\sigma_1, \ldots, \sigma_N; t)$. The probability per unit time that σ_i will flip is denoted by $w_i(\sigma_1, \ldots, \sigma_N)$. When the equilibrium probability distribution is described by a nearest-neighbor Hamiltonian, it suffices to let w_i depend also only on the nearest neighbors of σ_i. The probability of any configuration decreases because of the probability one of its spins may flip, but it grows because other configurations may decay to it. The transition rates into and out of a configuration are constrained by detailed balance. In one dimension, the equilibrium distribution is e^{-H}, with $H = -J\sum_i \sigma_i\sigma_{i+1}$ (k_BT has been adsorbed into the definition of J), and one has

$$\frac{\partial P}{\partial t}(\sigma_1, \ldots, \sigma_N; t) = -\sum_i w_i(\sigma_{i-1}, \sigma_i, \sigma_{i+1}) P(\sigma_1, \ldots, \sigma_N; t) + \sum_i w_i(\sigma_{i-1}, -\sigma_i, \sigma_{i+1}) \times P(\sigma_1, \ldots, -\sigma_i, \ldots, \sigma_N; t) \tag{1a}$$

and

$$\frac{w_i(\sigma_{i-1}, \sigma_i, \sigma_{i+1})}{w_i(\sigma_{i-1}, -\sigma_i, \sigma_{i+1})} = \exp[-2J\sigma_i(\sigma_{i-1} + \sigma_{i+1})] \ . \tag{1b}$$

Now define

$$\psi(\sigma_1, \ldots, \sigma_N; t) = e^{H/2} P(\sigma_1, \ldots, \sigma_N; t) \ .$$

Then ψ satisfies

$$\frac{\partial \psi}{\partial t}(\sigma_1, \ldots, \sigma_N; t) = -\sum_i w_i(\sigma_{i-1}, \sigma_i, \sigma_{i+1}) \psi(\sigma_1, \ldots, \sigma_N; t) + \sum w_i(\sigma_{i-1}, -\sigma_i, \sigma_{i+1}) \exp[-J\sigma_i(\sigma_{i-1} + \sigma_{i+1})] \times \psi(\sigma_1, \ldots, -\sigma_i, \ldots, \sigma_N; t) \ . \tag{2}$$

If ψ is thought of as a 2^N component vector, the matrix which describes its time development in (2) is Hermitian.

To make the analogy to quantum formalism, let

$$|\psi\rangle = \psi(\sigma_1^z, \ldots, \sigma_N^z; t) \sum_{\sigma_i^z = \pm 1} |\sigma_1^z\rangle \cdots |\sigma_N^z\rangle \ .$$

A state with σ_i^z flipped is described by σ_i^x acting on $|\psi\rangle$. If we take Glauber's form

$$w_i = \frac{\exp[-J\sigma_i^z(\sigma_{i-1}^z + \sigma_{i+1}^z)]}{\cosh J(\sigma_{i-1}^z + \sigma_{i+1}^z)} ,$$

then

$$\frac{\partial|\psi\rangle}{\partial t} = -H_Q|\psi\rangle ,$$

$$H_Q = \sum_i \frac{\exp[-J\sigma_i^z(\sigma_{i-1}^z + \sigma_{i+1}^z)] - \sigma_i^x}{\cosh J(\sigma_{i-1}^z + \sigma_{i+1}^z)} \quad (3)$$

is equivalent to (2). The eigenvalues of H_Q are all real and non-negative by stability. The ground state with energy zero is just

$$|\psi_0\rangle = e^{-H/2} \sum_{\sigma_i^z = \pm 1} |\sigma_1^z\rangle \cdots |\sigma_N^z\rangle .$$

Curiously enough, $\langle\psi_0|\psi_0\rangle$ is the partition function. An ostensibly simpler form of w_i, also consistent with (1b), is

$$w_i = \exp[-J\sigma_i^z(\sigma_{i-1}^z + \sigma_{i+1}^z)] ,$$

with

$$H_Q = \sum \exp[-J\sigma_i^z(\sigma_{i-1}^z + \sigma_{i+1}^z)] - \sigma_i^x . \quad (4)$$

Equations (3) and (4) are readily generalized to higher dimensions by replacing $\sigma_{i-1}^z + \sigma_{i+1}^z$ by a sum over the nearest neighbors of σ_i^z. Provided one rescales (4) by $\cosh(2J)$ in one dimension to make the overall frequency scales coincide with (3), one expects (3) and (4) to show the same critical behavior ($J \to \infty$ in one dimension).

In order to diagonalize (3) exactly, introduce bond variables $\tau_i^z = \sigma_i^z\sigma_{i+1}^z$. The action of σ_i^x on a state labeled by the τ^z variables is to flip both τ_{i-1}^z and τ_i^z; hence $\sigma_i^x = \tau_{i-1}^x\tau_i^x$. To complete the spin algebra, define $\tau_i^y = -i\tau_i^z\tau_i^x$. The τ variables on different sites commute and those on the same site obey the usual Pauli matrix relations. Equation (3) becomes, with these substitutions,

$$H_Q = \sum_i \Big[1 - \tanh(2J)\tau_i^z - \tfrac{1}{2}(\tau_i^x\tau_{i+1}^x + \tau_i^y\tau_{i+1}^y) + \frac{1}{2\cosh(2J)}(\tau_i^y\tau_{i+1}^y - \tau_i^x\tau_{i+1}^x)\Big] . \quad (5)$$

The well-known transcription of a one-dimensional spin problem into a fermion problem can now be applied to (5) and the fermion Hamiltonian diagonalized.[3] Each eigenvalue of (5) can be associated with a j particle state ($j = 0, 1, \ldots, N$) of the fermion problem with an energy given by the sum of j one-particle energies ϵ_n, with distinct values of k_n where

$$\epsilon_n = 2[1 - \tanh(2J)\cos(k_n)] . \quad (6)$$

The wave vector k_n takes on N uniformly distributed values in the range $-\pi \leq k_n < \pi$. Equation (6) together with the wave functions of the one-particle states were given by Glauber.[1] To compute response functions, one imagines that an ensemble with a slowly varying magnetic field, $he^{i\vec{k}_m\cdot\vec{R}_i}$ at R_i, has been established at $t = 0$. The field is turned off, and the system decays back to equilibrium. To first order in the magnetic field, the initial state is just proportional to

$$|\psi\rangle_{t=0} = \sum_i e^{i\vec{k}_m\cdot\vec{R}_i}\sigma_i^z|\psi_0\rangle ,$$

which is precisely the one particle eigenfunction found by Glauber. Its relaxation frequency is then given by (6) with $n = m$. For the uniform mode, the relaxation for $J \to \infty$ goes as $\sim[1 - \tanh(2J)] \sim \xi^{-2}$, where ξ is the correlation length.

The second form of Glauber's model (4) can also be simplified with the aid of bond variables. It becomes

$$H_Q = \sum_i [\cosh^2 J + \tau_i^z\tau_{i+1}^z \sinh^2 J - \tau_i^z \sinh(2J) - \tau_i^x\tau_{i+1}^x] . \quad (7)$$

Unfortunately (7) is not trivially solvable in the fermion representation. A number of other relaxation models can be reexpressed as quantum-mechanical systems. Kawasaki's model is rather complicated since the order parameter relaxes by spin exchange.[4] Another model is one in which individual spin flips are permitted subject to the constraint of energy conservation.[5] For a nearest-neighbor Hamiltonian in one dimension, the energy is the number of domain walls. Expressed in bond variables, the corresponding Hamiltonian is simply a spin-$\frac{1}{2}$ nearest-neighbor Heisenberg model with an additive constant that makes the ground state energy zero. The absence of a gap in the excitation spectrum is a consequence of the conservation law. To compute the long-time behavior of the site spin correlation functions seems rather awkward.

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[3] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) 16, 407 (1961).

[4] K. Kawasaki, Phys. Rev. 148, 375 (1966).

[5] This is model C of B. I. Halperin, P. C. Hohenberg, and S. K. Ma [Phys. Rev. B 10, 139 (1974)].