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Hydrodynamic Stability of an Evolving Vortex Ring in a Counterflow*

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The time-dependent motion of a vortex ring obeying the Hall–Vinen equations in a superfluid He counterflow is shown to be locally stable. Both the real and imaginary parts of the normal modes of oscillation scale as wave number squared.

1. INTRODUCTION

J. J. Thomson's calculation of the normal modes of a ring vortex has proved of more enduring value than W. Thomson's theory of vortex atoms, which was its motivation.[1,†] Vortex rings in an ideal fluid are neutrally stable and superimposed on their translational motion are a discrete set of oscillatory modes with a frequency proportional to $m(m^2-1)^{1/2}$ for integer m. The $m=0, 1$ modes correspond respectively to a uniform change in radius R and a rigid motion of the plane of the loop. Now in a counterflow experiment in superfluid ^{4}He as a consequence of the core–normal fluid drag, an isolated ring will retain its shape while its orientation and radius vary.[3,4] Its radius can increase no faster than linearly in time. The latter motion can be likened to a weak instability of the $m=0$ mode and one is lead to ask, in view of the close correspondence between vortex motion in an ideal fluid and a superfluid at sufficiently low temperatures, whether in a counterflow any of the $m>1$ modes might also be weakly unstable. We have addressed a somewhat more comprehensive problem, namely the stability of the time-dependent motion of an isolated vortex ring in a counterflow. The analysis in particular shows that the coupling between the $m=0, 1$ and higher modes does not induce any new instabilities.

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†Reference 1 was written without the benefit of vector calculus. Reference 2 is more useful.

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The stability of an evolving ring vortex is not unexpected. A continuum analysis of a rectilinear vortex parametrized with a line tension gives a damping (and frequency) proportional to the squared wave number. Furthermore, experiments on isolated vortices in He II have not suggested any new instabilities.[4]

The stability of growing vortex rings is also pertinent to the Iordanski–Langer–Fisher theory for the decay of a supercurrent.[5] It demonstrates that a ring, once nucleated, will grow until it reaches the walls if interactions with other vortices are neglected. If rings were unstable, one could imagine that they fragment into subcritical pieces, thereby suppressing nucleation. It is interesting to note that under certain conditions growing vapor bubbles are not stable, while drops growing in a vapor are.[6] Lastly, in working through the stability calculation we were able to clarify what, from our reading of the literature, appeared to be several minor ambiguities in the mathematical description of a time-evolving vortex.

2. ANALYSIS

Points $\mathbf{x}$ along the vortex will be labeled by either the arc length s or an angle θ measured from an origin at the ring center. The tangent $\mathbf{t}$, normal $\mathbf{n}$, and binormal $\mathbf{b}=\mathbf{t}\times\mathbf{n}$ constitute a right-handed orthonormal system of basis vectors at each point. The sense of $\mathbf{t}$ will coincide with the circulation. By definition, $d\mathbf{x}/ds=\mathbf{t}$ and $d\mathbf{t}/ds=\mathbf{n}/R$, where R is the local radius of curvature. For a circle, $d\mathbf{n}/ds=-\mathbf{t}/R$.

The velocity $\mathbf{v}_L$ of a point on the vortex is given implicitly by the equation[3]

$$-D\mathbf{t}\times[\mathbf{t}\times(\mathbf{v}_L-\mathbf{v}_n)]+D'\mathbf{t}\times(\mathbf{v}_L-\mathbf{v}_n)=\rho_s(h/m)\mathbf{t}\times(\mathbf{v}_L-\mathbf{v}_s) \quad (1)$$

where $\mathbf{v}_n$ and $\mathbf{v}_s$ are the normal and superfluid velocities, ρ_s is the superfluid density, and h/m is the quantum of circulation. Within the local induction approximation and neglecting boundaries,

$$\mathbf{v}_s=\mathbf{v}_s^0+(G/R)\mathbf{b} \quad (2)$$

where $\mathbf{v}_s^0$ is the spatially uniform background superfluid velocity.[4] The time dependence of $G=|G|=(\hbar/2m)[\ln(R/a)+O(1)]$ through R will be neglected. The core size is a.

The principal damping term in (1) is the first, which is customarily written as $D(\mathbf{v}_L-\mathbf{v}_n)$. It must, however, be perpendicular to $\mathbf{t}$, so a transverse projection operator was inserted.[7] Our equation for $\mathbf{v}_L$ differs from the conventional one only in terms proportional to $\mathbf{t}$.[3,8] Any such term can be removed by adjusting the labeling parameter, since motion of a vortex filament along itself is redundant. In particular, it does not correspond to any change in the many-particle He wave function. Note that (1) leaves the component of $\mathbf{v}_L$ parallel to $\mathbf{t}$ indeterminate. The constant D' is too small to measure, except possibly near T_λ.[8] Both D and D' are positive and vanish at $T=0$.

If terms explicitly proportional to $\mathbf{t}$ are omitted, then

$$\mathbf{v}_L-\mathbf{v}_n=\frac{1}{D^2+(D'-\rho_s h/m)^2}\left[\rho_s\frac{h}{m}\left(\rho_s\frac{h}{m}-D'\right)(\mathbf{v}_s-\mathbf{v}_n)+D\rho_s\frac{h}{m}\mathbf{t}\times(\mathbf{v}_n-\mathbf{v}_s)\right] \quad (3)$$

and at $T=0$, $\mathbf{v}_L=\mathbf{v}_s$. We henceforth measure all velocities relative to $\mathbf{v}_n$.* The line velocity is the total time derivative of $\mathbf{x}$. For this reason the arc length is an inconvenient parameter when the ring curvature changes in time. To simplify our notation we will rewrite (3) in terms of two new positive constants E and F, the first of which is nearly unity below 1.9K,[8]

$$\dot{\mathbf{x}}=\mathbf{v}_L=E\mathbf{v}_s-F\mathbf{t}\times\mathbf{v}_s \quad (4)$$

A dot will denote a time derivative.

To check that a vortex ring is a solution to (4) and to determine its time evolution we set

$$\mathbf{x}=\mathbf{x}_0-R_0\mathbf{n} \quad (5)$$

where $\mathbf{x}_0$ is the ring center, R_0 is its unperturbed radius and all quantities are functions of time. If (5) and (2) are substituted into (4), only $\mathbf{n}$ and $\mathbf{t}$ depend upon θ, and their integrals around the loop vanish by definition. Therefore, (4) breaks into two equations,

$$\dot{\mathbf{x}}_0=E[\mathbf{v}_s^0+(G/R_0)\mathbf{b}] \quad (6)$$

and

$$\dot{R}_0\mathbf{n}+\dot{\mathbf{n}}R_0=F(\mathbf{t}\times\mathbf{v}_s^0)-F(G/R_0)\mathbf{n} \quad (7)$$

Taking the dot product with $\mathbf{n}$ implies $(\mathbf{n}\cdot\dot{\mathbf{n}}=0)$

$$\dot{R}_0=-F(\mathbf{v}_s^0\cdot\mathbf{b}+G/R_0) \quad (8)$$

which is just the well-known result that vortices moving upstream with $R_0>G/|\mathbf{b}\cdot\mathbf{v}_s^0|$ will grow.[3] Note that the dissipative coefficients enter only the time scale. The components of (7) normal to $\mathbf{n}$ yield

$$\dot{\mathbf{n}}=(F/R_0)\mathbf{n}\cdot\mathbf{v}_s^0\mathbf{b} \quad (9a)$$

*We are assuming v_n is spatially uniform and have not explicitly noted that the mean phonon–roton velocity near the ring differs from v_n at infinity. Reference 7 shows this effect does not modify the form of (1) provided R exceeds the viscous penetration depth and the mean free path in the normal fluid.

If the ring rotates rigidly, (9a) is equivalent to rotation about an axis parallel to $\mathbf{v}_s^0 \times \mathbf{b}$ through the center of the ring with an angular velocity $F|\mathbf{v}_s^0 \times \mathbf{b}|/R_0$. The axis of rotation, without loss of generality, may be taken perpendicular to $\mathbf{b}$. For the vectors $\mathbf{t}$ and $\mathbf{b}$ one has

$$\dot{\mathbf{t}} = (F/R_0)\mathbf{t} \cdot \mathbf{v}_s^0 \mathbf{b} \tag{9b}$$

$$\dot{\mathbf{b}} = -(F/R_0)(\mathbf{v}_s^0 - \mathbf{b} \cdot \mathbf{v}_s^0 \mathbf{b}) \tag{9c}$$

If φ is the angle between $\mathbf{b}$ and $\mathbf{v}_s^0$, then,

$$\dot{\varphi} = (F/R_0) v_s^0 \sin \varphi \tag{10}$$

Both $\varphi = 0$ and $\varphi = \pi$ are stationary solutions of (10), but only the latter is stable. The plane of the ring rotates to make $\mathbf{b}$ antiparallel to $\mathbf{v}_s^0$.

To examine the stability of the solution represented by (8) and (9) to (4), we assume

$$\mathbf{x} = \mathbf{x}_0 - R_0 \mathbf{n} + \alpha \mathbf{t} + \beta \mathbf{n} + \gamma \mathbf{b} \tag{11}$$

and work to first order in the small quantities α, β, and γ, each assumed of order ε and an arbitrary function of time and θ. All other quantities, in particular $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$, are given by the unperturbed solution. The length scale is set by R_0 given in (8). The term proportional to $\mathbf{t}$ has been included only in order to check that α does not enter the equations for $\dot{\beta}$ and $\dot{\gamma}$.

To compute the change of (4) to first order in ε. it is convenient to rewrite (2) in invariant form:

$$\mathbf{v}_s = \mathbf{v}_s^0 + G \frac{d\mathbf{x}}{ds} \times \frac{d^2\mathbf{x}}{ds^2} \tag{12}$$

and change the dependent variable to the angle θ,

$$s' = R_0 + \alpha' - \beta + \varphi(\varepsilon^2) \tag{13}$$

Derivatives with respect to θ will be indicated by a prime. Substituting yields

$$d\mathbf{x}/ds = \mathbf{t} + (\alpha + \beta')\mathbf{n}/R_0 + \gamma' \mathbf{b}/R_0 + \varphi(\varepsilon^2) \tag{14}$$

and

$$\mathbf{v}_s = \mathbf{v}_s^0 + G\mathbf{b}/R_0 - G\gamma'\mathbf{t}/R_0^2 - G\gamma''\mathbf{n}/R_0^2 + G(\beta' + \beta)\mathbf{b}/R_0^2 + \varphi(\varepsilon^2) \tag{15}$$

Substituting (14) and (15) into the right-hand side of (4), (11) into the left-hand side, and remembering to take the time derivatives of $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ yields for the coefficients of $\mathbf{n}$ and $\mathbf{b}$

$$R_0^2 \dot{\beta} = -EG\gamma'' + FG(\beta + \beta'') + FR_0(\gamma \mathbf{n} \cdot \mathbf{v}_s^0 - \gamma' \mathbf{t} \cdot \mathbf{v}_s^0) \tag{16a}$$

and

$$R_0^2\dot{\gamma} = EG(\beta+\beta'') + FG\gamma'' + FR_0(\beta'\mathbf{t}\cdot\mathbf{v}_s^0 - \beta\mathbf{n}\cdot\mathbf{v}_s^0) \tag{16b}$$

The equation for $\dot{\alpha}$ corresponds to the coefficients of $\mathbf{t}$, which may be transformed away by adding an $\theta(\varepsilon)$ multiple of (14) to the right-hand side of (4). Note that α does not appear in (16).

For $F=0$ and $E=1$ ($T=0$), the first term of (16) yields Thomson's result,

$$\omega^2 = G^2m^2(m^2-1)/R_0^4 \tag{17}$$

for integer m. For F nonzero, R_0 and possibly $\mathbf{t}$ and $\mathbf{n}$ are time dependent.

Consider first the case $\mathbf{b}$ parallel or antiparallel to $\mathbf{v}_s^0$, which eliminates the third term in (16). The $m=0$, or θ-independent, mode may be computed directly from (6) and (8) by replacing R_0 by $R_0-\beta$ and expanding in β. This mode simply corresponds to a change of R_0. The $m=1$ mode is no longer trivial since $\mathbf{v}_s^0$ establishes a preferred direction.

To complete the solution for $\mathbf{t}\cdot\mathbf{v}_s^0=\mathbf{n}\cdot\mathbf{v}_s^0=0$, define a new time variable τ as a monotone increasing function of the time,

$$d\tau = dt/R_0^2$$

or,

$$\tau = \frac{1}{FG}\ln\left|\frac{R_0 \pm R^*}{R_0}\right| \tag{18}$$

An additive constant has been omitted in (18). The $\pm$ signs refer respectively to $\mathbf{b}$ parallel and antiparallel to $\mathbf{v}_s^0$, and $R^*=G/|v_s^0|$. Assuming $\beta, \gamma \sim e^{-\lambda\tau}$,

$$\lambda_{1,2} = FG(m^2-1/2) \pm i[E^2G^2m^2(m^2-1) - F^2G^2(m^2-1/2)^2]^{1/2} \tag{19}$$

so that $\mathrm{Re}\,\lambda_{1,2}/FG > 1$ for $m>1$. Equation (19) then implies

$$\beta, \gamma \sim (R_0/|R_0 \pm R^*|)^{\lambda_{1/2}/FG} \tag{20}$$

When $m=1$ one finds

$$d\gamma/d\tau = -FG\gamma, \qquad d\beta/d\tau = EG\gamma \tag{21}$$

The $m=1$ modes correspond to rigid motions of the ring whose precise form is given below. For $\mathbf{v}_s^0$ parallel to $\mathbf{b}$[+in (20)], R_0 decreases to zero along with the $m>1$ components of β/R_0 and γ/R_0.

With the direction of $\mathbf{b}$ reversed [$-$ in (20)], R_0 increases to infinity if it is greater than R^* initially, and β/R_0 and γ/R_0 then decrease to zero. When

$R_0 < R^*$ initially, it decreases to zero along with β and γ. If $R_0 = R^*$, then R_0 remains time independent and $\beta, \gamma \sim \exp(-\lambda_{1,2}t)$.

For the general case of $\mathbf{b} \cdot \mathbf{v}_s^0$ arbitrary, it is convenient to chose θ such that $\mathbf{t}(0) \cdot \mathbf{v}_s^0 = 0$ and define

$$\begin{aligned} \mathbf{t}(\theta) &= \cos\theta\, \mathbf{t}(0) + \sin\theta\, \mathbf{n}(0) \\ \mathbf{n}(\theta) &= -\sin\theta\, \mathbf{t}(0) + \cos\theta\, \mathbf{n}(0) \end{aligned} \tag{22}$$

Only $\mathbf{n}(0)$ is time dependent $[\mathbf{v}_s^0 \cdot \mathbf{n}(0) = |v_s^0| \sin\varphi$ using (10)], since $t(0)$ is the axis about which the loop rotates. Rewriting (16)

$$R_0^2 \dot{\beta} = -EG\gamma'' + FG(\beta + \beta'') + FG(R_0/R^*) \sin\varphi\, (\gamma \cos\theta - \gamma' \sin\theta) \tag{23a}$$

$$R_0^2 \dot{\gamma} = EG(\beta + \beta'') + EG\gamma'' + FG(R_0/R^*) \sin\varphi (\beta' \sin\theta - \beta \cos\theta) \tag{23b}$$

The physically relevant piece of the R_0–φ plane, $0 \le \varphi \le \pi$, $R_0 \ge 0$, is divided into two regions according to whether $\lim_{t\to\infty} R_0 = 0$ or ∞. In the latter case φ approaches π and (10) can be linearized about this point. Setting $\varphi = \pi$ in (8) permits an explicit solution for φ as a function of R_0,

$$\varphi = \pi - c/|R_0 - R^*|$$

where c is a positive constant. As $R_0 \to \infty$, the factor $R_0 \sin\varphi$ in (23a) and (23b) approaches a constant. The time variable can again be eliminated in favor of τ defined in (18). Now an infinite range of R_0 maps out a finite range of τ, and irrespective of the eigenvalues of (23), β and γ will remain finite as $R_0 \to \infty$.

For the collapsing ring, φ may not reach π by the time $R_0 = 0$. Changing from time to τ, $R_0 \sim e^{-\tau}$ and τ runs to infinity as R_0 decreases to zero. When $R_0/R^* \ll 1$, the last terms in (23a) and (23b) lead to a narrow band of eigenstates centered about each of the discrete eigenmodes (19). The modes that correspond to $m > 1$ are all stable since $\mathrm{Re}(\lambda_{1,2})/FG > 1$. The $m = 0, 1$ modes decouple from the rest and correspond to various rigid motions of the vortex ring without change of shape. The two modes with $\beta, \gamma \sim \sin\theta$ decouple further and satisfy (21). They correspond respectively to a rigid translation along $\mathbf{n}(0)$ (frequency zero) and a rotation about an axis parallel to $\mathbf{n}(0)$ through the center of the ring [see (22)]. The remaining four modes with β, γ either constant or proportional to $\cos\theta$ correspond respectively to a charge in radius, a translation parallel to $\mathbf{b}$, a translation along $\mathbf{t}(0)$, and a rotation about $\mathbf{t}(0)$ [i.e., $\mathbf{t}(0) \times \mathbf{n}(\theta) = (\cos\theta)\mathbf{b}$].

Finally, if the initial values of R_0 and φ are such as to make $R_0 \to R^*$ as $t \to \infty$, then φ approaches π and linearizing (10) yields

$$\varphi = \pi - c \exp\left(-Fv_s^0 t/R^*\right)$$

Thus the last terms in (23a) and (23b) again tend to zero, and the above analysis with τ replaced by t may be repeated.

3. CONCLUSION

We have demonstrated the linear stability of a vortex ring evolving in a superfluid counterflow within the local induction approximation. Corrections to the local induction approximation are a factor of $\ln(R_0/a)$ smaller than the last term in (2), and for R_0/a sufficiently large cannot reverse the sign of the damping of the $m > 1$ modes. in this context the work of Grant should be cited; he calculated the excitation spectrum of a vortex ring at zero temperature from the Gross–Pitaevskii equation.[9] He did not consider the hydrodynamical problem.

Recent work in an ideal fluid by Widnall and others has revealed a number of instabilities of finite core vortices.[10] We believe that the unstable modes are confined to large wave numbers, $ka \sim 1$, where in the helium problem quantum mechanical effects would be important. We are not aware of any deficiencies in the vortex filament idealization when applied to long-wavelength modes.

Hydrodynamic damping, as we have seen, tends to suppress high wave numbers. Numerical simulations of a vortex tangle in helium for $T > 0$ with a Biot-Savart code would be better controlled than in an ideal fluid where high-wavenumber noise can contaminate the larger scales.[11] Note should also be taken in the helium literature of the smoke ring experiments of Kambe and Takao, who observed a ring collision.[2] Somewhat surprisingly, the two rings combined and separated each more or less intact. The behavior of finite-amplitude perturbations to a vortex ring in helium remains an open question. Numerical experiments in an ideal fluid have shown how perturbations on a smooth section of line rapidly grow, twist up, and possibly, in a real fluid, pinch off, shortening the line.[11] The numerical codes go beyond the local induction approximation within which, as Hasimoto has shown, the evolution of a vortex line is isomorphic to the nonlinear Schrödinger equation.[12]

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REFERENCES

1. J. J. Thomson, *A Treatise on the Motion of Vortex Rings* (MacMillan, London, 1883; reprinted Dawsons, London, 1968).
2. T. Kambe and T. Takao, *J. Phys. Soc. Japan* **31**, 591 (1971).
3. W. F. Vinen, in *Progress in Low Temperature Physics* **3**, C. J. Gorter, ed. (North-Holland, Amsterdam, 1961), and references therein.
4. A. L. Fetter, in *The Physics of Liquid and Solid Helium*, Vol. 1, K. H. Bennemann and J. B. Ketterson, eds. (Wiley, New York, 1976).
5. J. S. Langer and J. Reppy, in *Progress in Low Temperature Physics* **6**, C. J. Gorter, ed. (North-Holland, Amsterdam, 1970).
6. M. S. Plesset, *J. Appl. Phys.* **25**, 96 (1954).
7. H. E. Hall and W. F. Vinen, *Proc. Roy. Sci. A***238**, 215 (1956).
8. L. J. Cambell, *J. Low Temp. Phys.* **3**, 175 (1970).
9. J. Grant, *J. Phys. A***4**, 695 (1971).
10. S. Widnall and C. Y. Tsai, *Phil. Trans. R. Sci. Lond.* **283**, 273 (1977) and references therein.
11. A. Leonard, in *Springer Lecture Notes in Physics, No* 35 (1975), p. 245; in *Springer Lecture Notes in Physics*, No. 59 (1977), p. 280; and private communication.
12. H. Hasimoto, *J. Fluid Mech.* **51**, 477 (1972).