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Two-scale-factor universality and the renormalization group

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The hypothesis of two-scale-factor universality, originally proposed by Stauffer, Ferer, and Wortis, is shown to follow from the renormalization-group approach, for systems close to their critical point. Values of the universal ratios involving correlation length and specific-heat amplitudes are obtained from the ϵ expansion, for Ising, X-Y, and Heisenberg models. In the latter two cases the correlation function has a power-law behavior at large distances below T_c, and the (transverse) correlation length is defined in terms of the stiffness constant ρ_s. Experimental values of the correlation lengths and amplitude ratios are determined for superfluid ^{4}He, which is X-Y-like, and for the Heisenberg antiferromagnet $RbMnF_3$. Comparisons are made between the values of the amplitude ratios coming from ϵ expansions, series, and experiments.

I. INTRODUCTION

The phenomenological theory of *scaling*[1] has been extremely useful for understanding critical phenomena in model systems and real materials. A related concept, formulated as the hypothesis of *universality*,[2] greatly reduces the variety of different types of critical behavior, by dividing all systems into a small number of equivalence classes. Within each class the exponents and the equation of state will be the same, provided one fixes the scales of the order parameter and its conjugate field appropriately. Thus, apart from two scale factors which will differ from system to system, the thermodynamic functions of all elements in the same class will be identical, sufficiently close to the critical point.

The scaling hypothesis was extended to time-independent correlations of the order parameter in the earliest formulations,[1] and it was found[1,3] that the correlation exponents were simply related to the thermodynamic ones. A few years ago, Stauffer, Ferer, and Wortis[4] generalized the concept of thermodynamic universality, by the hypothesis of *two-scale-factor universality* for correlation functions. This hypothesis states that the correlation function for a system is fully determined near the critical point once the two independent thermodynamic scales have been chosen. This means that the length scale is not independent, but is universally related to the thermodynamic scales. Stauffer, Ferer, and Wortis showed that the hypothesis was exactly satisfied in the two-dimensional Ising model, and held rather accurately in a number of other models.

The *renormalization-group* approach[5] provides a systematic method for proving scaling relations, and for calculating exponents and universal amplitude ratios as expansions in $\epsilon = 4 - d$ or $1/n$ (d is the dimensionality, and n the number of components of the order parameter). Indeed, early applications of Wilson's expansion methods were the calculation of the universal equation of state[6,7] and of the universal order-parameter–order-parameter correlation function,[8,9] and the verification of two-scale-factor universality to second order in ϵ.[10]

In the present paper, we pursue this program by proving two-scale-factor universality to all orders in ϵ, by completing the order ϵ^2 calculation, and by extending the calculation of the corresponding amplitude ratios to systems with continuous symmetry. In these systems, the correlation function in the ordered state falls off as a power of the distance, rather than exponentially, which necessitates a somewhat different treatment than the one for Ising systems. As was first pointed out by Josephson,[11] an interesting application is to superfluid helium, where the superfluid density yields a direct measure of the transverse correlation length below T_λ. A similar system is the isotropic antiferromagnet $RbMnF_3$, where the spin waves in the ordered state may also be related to the transverse correlation length. Following a suggestion of Ferer,[12] we are thus able to obtain, for such systems, experimental estimates of the universal amplitude ratio connecting the correlation length to the singular part of the specific heat. The value of this ratio may be compared to an extrapolation to $d=3$ of the ϵ expansion. Using this extrapolated value and the measured specific heat, we can predict values for the transverse correlation length, and we find in each case that the theory overestimates the correlation length by roughly 50%, which may be within the error bars of the extrapolation of the ϵ expansion. In liquid helium above T_λ we may use the series value of the amplitude ratio and the measured specific heat to obtain a value of the correlation length which is otherwise unmeasurable.

In Sec. II the appropriate amplitude ratios are defined, first for the case of exponentially decaying correlations, and then for power-law correlations. The results of the ϵ expansion are given to second order, and extrapolated values for $d=3$ obtained. Series values for $T>T_c$ and for $n=1$ ($T<T_c$) are used to estimate the accuracy of these extrapolations. Section III discusses the applications to superfluid helium and $RbMnF_3$. The proof of two-scale-factor universality from renormalization-group recursion relations, and the details of the ϵ expansion are contained in the appendixes.

II. THEORETICAL ASPECTS

A. Systems with exponentially decaying correlations

We first discuss the situation for $T>T_c$, since it is generally simpler. In order to formulate the statement of two-scale-factor universality, we have to start by defining the scale factors. In doing so, we follow the approach of Ref. 10, which is somewhat different from that adopted by Betts, Guttmann, and Joyce,[2] or by Stauffer, Ferer, and Wortis.[4] We start by considering the wave-vector-dependent susceptibility $\hat{\chi}(q,t,M)$, which is the Fourier transform of

$$\chi(\vec{r},t,M) = (nk_BT)^{-1}\sum_{\alpha=1}^{n}\left[\langle\psi_\alpha(\vec{r})\psi_\alpha(0)\rangle - \langle\psi_\alpha(\vec{r})\rangle\langle\psi_\alpha(0)\rangle\right]. \quad (1)$$

Here, $t=(T-T_c)/T_c$, M is the "magnetization" (or, generally, the average of the order parameter,[13] $\langle\psi_\alpha\rangle = M\delta_{\alpha,1}$), and $\psi_\alpha(\vec{r})$ is the αth component of the n-component order parameter at the site $\vec{r}$. The *scaling assumption*[1] states that for $q,t,M\to 0$, $\hat{\chi}$ may be written in the form

$$\hat{\chi}(q,t,M) = t^{-\gamma}Z(tM^{-1/\beta}, t^{-\nu}q), \quad (2)$$

where γ, β, and ν are the usual critical exponents.[14] Clearly, the relation

$$z=Z(x,y) \quad (3)$$

involves three scales, i.e., those of the variables x, y, and z. Given the scaling form (2), it is natural to make a *universality hypothesis* (*three*-scale universality): Once the scales of x, y, and z are chosen in a unique way (so that the function Z or its derivatives assume three given boundary values), then the resulting function is the same for all systems within a "universality class." The further hypothesis of *two-scale-factor universality* says, in fact, that only two of the three constants are independent.

The two purely thermodynamic scales may be defined by considering the function $\hat{\chi}(0,t,M)$, which is directly related to the equation of state, assumed to have the scaling form[15]

$$H/M^\delta = h(t/M^{1/\beta}), \quad (4)$$

where H is the ordering field conjugate to M. The scale of $t/M^{1/\beta}$ is determined by the coexistence curve,

$$t/M^{1/\beta} = -x_0, \quad H=0, \quad (5)$$

and the scale of $h(x)$ is determined by the critical isotherm,

$$H/M^\delta = h_0, \quad t=0. \quad (6)$$

Rescaling H/M^δ by h_0 and $t/M^{1/\beta}$ by x_0 we obtain a universal scaling function,[16]

$$\tilde{h}(\tilde{x}) = \tilde{h}(x/x_0) = h_0^{-1}h(x). \quad (7)$$

The nonuniversal constants x_0 and h_0 then determine all the thermodynamic critical amplitudes. For example, for $t>0$, $H=0$, and $q=0$,

$$\hat{\chi}(0,t,0) = \Gamma t^{-\gamma}, \quad (8)$$

with[17]

$$\Gamma = x_0^\gamma h_0^{-1}\lim_{\tilde{x}\to\infty}\frac{\tilde{x}^\gamma}{\tilde{h}(\tilde{x})}. \quad (9)$$

Similarly, the singular term in the specific heat per unit volume above T_c is given by[18]

$$C_+^s = (A/\alpha)t^{-\alpha} \quad (H=0,\ t>0), \quad (10)$$

with A uniquely determined by x_0, h_0, and $\tilde{h}(\tilde{x})$. Equation (9) determines the value of $Z(\infty,0)$, i.e., the scale of z in Eq. (3). To fully characterize $Z(x,y)$ we now need only one more boundary condition. Following Ref. 8, we choose this to be determined by the small-y behavior of $Z(0,y)$ which, for an exponentially decaying correlation function, may be written as

$$Z(0,y) = \Gamma[1+(\xi_0 y)^2 + O((\xi_0 y)^4)]^{-1}. \quad (11)$$

This equation defines the amplitude of the *correlation length*,

$$\xi_+ = \xi_0 t^{-\nu} \quad (t>0,\ H=0), \quad (12)$$

which is related to the second moment of the correlation function at zero field above T_c.

Returning to (2), we can now define

$$\tilde{Z}(\tilde{x}, \tilde{y}) = \tilde{Z}(x/x_0, \xi_0 y) = Z(x, y)/\Gamma , \quad (13)$$

and $\tilde{Z}(\tilde{x}, \tilde{y})$ is a universal function, just as the exponents are.

Betts, Guttmann, and Joyce[2] and Stauffer, Ferer, and Wortis[4] also define three scale factors, i.e., l, n, and g, related to the scales of length, ordering field, and temperature. The basic conceptual difference between our formulation and theirs is that their scale factors are defined as ratios of the normalization constants (e.g., x_0, h_0, and ξ_0) for a given system to their counterparts in a reference system. We feel that the present absolute definitions are more useful for experimental analysis.

We are now ready to state the hypothesis of *two-scale-factor universality*[4]: Only two of the three normalization constants x_0, h_0, and ξ_0 are independent, and the three constants are related through the universality of the combination[19]

$$R_\xi^+ \equiv \xi_+ (\alpha t^2 C_+^s)^{1/d} , \quad (14)$$

where d is the dimensionality of the system. Note that the temperature independence of R_ξ^+ is just a consequence of the scaling law

$$d\nu = 2 - \alpha . \quad (15)$$

The quantity R_ξ^+ may also be expressed as a universal combination of critical amplitudes,

$$R_\xi^+ = A^{1/d} \xi_0 . \quad (16)$$

In Appendix A we show that the universality of R_ξ^+ follows from the assumptions of the renormalization-group approach,[5] in much the same way as the scaling relation (15).

If the constant R_ξ^+ can be computed theoretically or obtained by measurements for one system in a universality class, then the relation (14) enables one to predict the correlation length, for any other member of the class, once the specific heat is accurately measured. For $t \to 0^+$, we have

$$\xi_+(t) \underset{t\to 0^+}{\simeq} R_\xi^+ [\alpha t^2 C_+^s(t)]^{-1/d} . \quad (17)$$

The situation for the Ising model at zero field below T_c ($n = 1$) is very similar: The wave-vector-dependent susceptibility is now

$$\hat{\chi}(q, t, M) = t^{-\gamma'} Z(-x_0, |t|^{-\nu'} q) , \quad (18)$$

and for small $|t|^{-\nu'} q$ it can be written as

$$\hat{\chi}(q, t, M) = \Gamma' |t|^{-\gamma'} [1 + (\xi_0' |t|^{-\nu'} q)^2 + O((\xi_0' |t|^{-\nu'} q)^4)]^{-1} . \quad (19)$$

From scaling we have

$$\nu = \nu' , \quad \gamma = \gamma' . \quad (20)$$

Universality of $\tilde{Z}(\tilde{x}, \tilde{y})$ leads to universality of Γ/Γ' and of ξ_0/ξ_0'. Similarly, the singular term in the specific heat per unit volume below T_c at $H = 0$ is given by[18]

$$C_-^s = (A'/\alpha') |t|^{-\alpha'} \quad (H = 0, \ t < 0) , \quad (21)$$

with $\alpha = \alpha'$, and with a universal ratio A/A'.

In analogy to (14), we can thus define

$$R_\xi^- = \xi_- (\alpha t^2 C_-^s)^{1/d} , \quad (22)$$

and this is directly related to R_ξ^+ through

$$R_\xi^+ / R_\xi^- = (A/A')^{1/d} (\xi_0 / \xi_0') . \quad (23)$$

The quantities R_ξ^+ and R_ξ^- were evaluated, for a number of models and real materials, in the original work of Stauffer, Ferer, and Wortis.[4] These ratios were generally found to be independent of lattice structure and spin value for the series expansions,[20] but were much less regular for the experimental data,[21] as is indeed the case for the exponents themselves. In view of the rather large experimental uncertainties involved in determining the specific heat and correlation-length singularities in real materials, we do not consider the present disagreement with two-scale-factor universality to be conclusive, but rather an indication that data on critical amplitudes may not be quantitatively reliable. A similar situation exists for purely thermodynamic amplitude ratios.[22]

The universality of the combination (14) was checked by one of us,[10] to second order in $\epsilon = 4 - d$, using Feynman-graph expansions based on the renormalization group.[5] An expression was found for $h_0 x_0^{\alpha-2} \xi_0^d = (R_\xi^+)^d / \tilde{A}$, where $\tilde{A}$ is a universal quantity, defined in Eq. (B10), for short-range n-component models. Results depended only on d and on n, and not on other properties of the starting model, such as the cutoff Λ or the four-spin coupling constant u_0.[23] Comparison of the result of Ref. 10 with experiments or with models was complicated, however, because $\tilde{A}$ could only be roughly estimated and because the ϵ expansion calculation was difficult to extrapolate to $\epsilon = 1$. More recently, two of the present authors[24] used a direct renormalization-group recursion-relation approach[25] to calculate R_ξ^+ to lowest order in ϵ in $d = 4 - \epsilon$ dimensions. A closely related constant was calculated exactly for short-range models at $d = 4$ as well as for the dipolar Ising model at $d = 3$. This approach can be generalized to show that under the assumptions of the renormalization group,[5] R_ξ^+ (and R_ξ^-) are indeed universal. Since this proof is rather technical, we present it in Appendix A. The alternative approach, used in Ref. 10, which employs Wilson's Feynman-graph expansion,[5] is more directly applicable to an explicit calculation of the ϵ expansion of R_ξ^+. This calculation is outlined in Appendix B. The final result is

$$(R_\xi^+)^d = \tfrac{1}{4} n K_d \left[1 + \epsilon \left(1 - \frac{9}{n+8}\right) + O(\epsilon^2)\right], \quad (24)$$

where

$$K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2). \quad (25)$$

To obtain R_ξ^- for the Ising model, one needs A/A' and ξ_0/ξ_0' [see Eq. (23)]. These ratios were calculated by Brézin, Le Guillou, and Zinn-Justin,[26] with the results

$$A/A' = 2^{\alpha-2}(1+\epsilon)n + O(\epsilon^2) \quad (26)$$

and, for $n=1$,

$$\xi_0/\xi_0' = 2^{\nu}(1 + \tfrac{5}{24}\epsilon + 0.0176\epsilon^2) + O(\epsilon^3). \quad (27)$$

(For the ϵ expansions of α, ν, etc., see, e.g., Ref. 5.) A simple substitution into Eq. (23) thus yields

$$(R_\xi^-)^d = \tfrac{1}{4} K_d(1 - \tfrac{11}{6}\epsilon) + O(\epsilon^2), \quad n = 1. \quad (28)$$

The extrapolation of the ϵ expansion (28) [just as that of[22] (26)] to $\epsilon = 1$ is ambiguous, owing to the large negative coefficient of ϵ. The uncertainties involved may be appreciated by considering the spherical model, which is the $n \to \infty$ limit of the n-component models.[27] For this model, Gerber[28] calculated R_ξ^+ exactly, with the result

$$\frac{(R_\xi^+)^d}{n} \xrightarrow[n\to\infty]{} \frac{K_d(2-\epsilon)^{-2}\pi\epsilon/2}{\sin(\pi\epsilon/2)}. \quad (29)$$

At $d=3$, this yields $(R_\xi^+)^d/n \to \frac{1}{2}\pi K_3 = 1/4\pi = 0.0796$, or $R_\xi^+/n^{1/3} \to 0.43$. An ϵ expansion of (29) gives, in agreement with (24),

$$(R_\xi^+)^d/n \to \tfrac{1}{4}K_d(1+\epsilon) + O(\epsilon^2). \quad (30)$$

Using $K_3 = 1/2\pi^2$ and $\epsilon = 1$, (30) yields $R_\xi^+/n^{1/3} \to 0.29$. A slightly better agreement with the exact result is obtained if we replace (30) by

$$R_\xi^+/n^{1/d} \to (\tfrac{1}{4}K_d)^{1/d}(1+\epsilon/d) + O(\epsilon^2), \quad (31)$$

and now extrapolate to $\epsilon = 1$, $d = 3$. This yields $R_\xi^+/n^{1/d} \to 0.31$. From now on, the procedure indicated in Eq. (31) will be used for extrapolating ratios such as R_ξ^+ and R_ξ^- to $\epsilon = 1$.

Fortunately, the coefficient of ϵ in Eq. (24) is not very large, so that extrapolation of this expansion to $\epsilon = 1$ is expected to be more reliable. [This line of reasoning is of course only suggestive, since Eq. (24) has a vanishing coefficient of ϵ for $n=1$, but the resulting extrapolation $R_\xi^+ = 0.23$ still differs somewhat from the series value $R_\xi^+ = 0.26$.] Our extrapolations for R_ξ^+ and R_ξ^- from the ϵ expansion are compared with series values in Table I. The ϵ-expansion values seem to be consistently smaller than the series values, by ~20% for R_ξ^+ and by a factor of 2 for R_ξ^- ($n=1$). This latter result is not surprising, however, owing to the large negative coefficient of ϵ in Eq. (28). If the extrapolated ϵ-expansion value of R_ξ^+ is combined with an experimental measurement of C_+^s, to predict ξ_+ [see (17)], then ξ_+ will be underestimated by ~20%. This comparison with the available series data may serve as an estimate of the ambiguities

TABLE I. Model values of universal amplitude ratios, at $d=3$,

	$n=1$		$n=2$		$n=3$	
	Series	ϵ expansion	Series	ϵ expansion	Series	ϵ expansion
R_ξ^+	0.26[a]	0.23[b]	0.36[c]	0.30[b]	0.42[a]	0.36[b]
A/A'	0.51[d]	0.55[e]	1.08[d]	0.99[e]	1.52[d]	1.36[e]
ξ_0/ξ_0'	1.96[f]	1.91[e]	⋯	⋯	⋯	⋯
R_ξ^-	0.17[g]	0.09[c]	⋯	⋯	⋯	⋯
ξ_0/ξ_0^T	⋯	⋯	⋯	0.27[h]	⋯	0.30[h]
R_ξ^T	⋯	⋯	⋯	0.95[i]	⋯	0.88[i]

[a] From Ref. 4.
[b] ϵ expansions (24) and (28) extrapolated as explained in Eq. (31).
[c] From Ferer, Moore, and Wortis, Ref. 20.
[d] Obtained from the relation $A/A' \approx 1 - 4\alpha$, with α taken from series. See Refs. 16 and 22.
[e] From Ref. 26.
[f] From Tarko and Fisher, Ref. 3
[g] From Eq. (23).
[h] From Eq. (43), with $d=3$, $\epsilon=1$, and $1-2\beta = 3\epsilon/(n+8)$.
[i] From Eq. (44), with $d=3$, $\epsilon=1$.

involved in the extrapolation of the ϵ expansion. If we include the comparison in the case of the spherical model, we are led to an error estimate of order 20%–50%. Such estimates are useful, since in the following discussion we shall extrapolate ϵ expansions of amplitude ratios, for which no series values are available.

B. Systems with power-law correlations

For $n>1$ (e.g., X-Y and Heisenberg systems), the correlations below T_c do not decay exponentially, as would follow from Eq. (19), but rather according to a power law, whose form may be determined from hydrodynamics,[29] or from the renormalization group.[6,7,30]

For $n>1$ and $T<T_c$, we must define two response functions, i.e., $\hat{\chi}_L$, which is the Fourier transform of the longitudinal correlation function

$$\chi_L(\vec{r},t,M)=(k_BT)^{-1}(\langle\psi_1(\vec{r})\psi_1(0)\rangle-M^2), \quad (32)$$

and $\hat{\chi}_T$, the Fourier transform of

$$\chi_T(\vec{r},t,M)=(k_BT)^{-1}\langle\psi_\alpha(\vec{r})\psi_\alpha(0)\rangle, \quad \alpha\neq 1. \quad (33)$$

Clearly,

$$\hat{\chi}(\vec{q},t,M)=[\hat{\chi}_L+(n-1)\hat{\chi}_T]/n. \quad (34)$$

For $H=0$ and $t<0$, $\hat{\chi}$ is dominated at long wavelengths by $\hat{\chi}_T$, which has the form

$$\hat{\chi}_T(q,t,M)\approx M^2/k_BT\rho_s q^2, \quad q\to 0,\ t<0,\ H=0 \quad (35)$$

where ρ_s is a stiffness constant, which depends on temperature. Equation (35) was derived in Ref. 29 for the usual three-dimensional case, but it holds also for any d at which long-range order exists. The coordinate-space transverse response function is then

$$\chi_T(\vec{r},t,M)\approx A_d M^2/k_BT\rho_s r^{d-2}, \quad r\to\infty,\ t<0,\ H=0 \quad (36)$$

where the coefficient

$$A_d=\Gamma(d/2)[\pi^{d/2}(2d-4)]^{-1}$$

comes from the Fourier transform of q^{-2} in d dimensions. Since χ, as defined in (1), has the dimensions of $(k_BT)^{-1}M^2$, a (transverse) correlation length ξ_T can be defined by the relation[31-35]

$$\chi_T(\vec{r},t,M)=A_d' M^2(k_BT)^{-1}(\xi_T/r)^{d-2}, \quad (37)$$

where A_d' is a numerical constant, which may be chosen to have any fixed positive value. By convention,[36] we shall choose $A_d'=A_d$, so that Eq. (35) reads

$$\hat{\chi}_T(q,t,M)\approx M^2\xi_T^{d-2}/k_BTq^2, \quad q\to 0,\ t<0,\ H=0 \quad (38)$$

and[36]

$$\xi_T^{2-d}=\rho_s. \quad (39)$$

The length ξ_T was denoted κ_-^{-1} in Ref. 35. In superfluid helium the stiffness constant ρ_s is related to the usual superfluid density $\tilde{\rho}_s$, in units of mass per unit volume, by the equation[37]

$$\rho_s=(\hbar^2/m_{\rm He}^2k_BT)\tilde{\rho}_s, \quad (40)$$

where $m_{\rm He}$ is the helium mass. According to the scaling hypothesis[1,11] any other relevant correlation length for the system below T_c must be a multiple of the length ξ_T defined in (39). Moreover, the exponent ν' for ξ_T must be equal to ν,

$$\xi_T=\xi_0^T(-t)^{-\nu'}=\xi_0^T(-t)^{-\nu}. \quad (41)$$

The two-scale-factor universality assumption below T_c, first formulated for helium by Ferer,[12] may now be stated analogously to (22), in terms of

$$R_\xi^T=\xi_T(\alpha t^2C_-^s)^{1/d}, \quad (42)$$

where ξ_T is defined by (39). The quantity R_ξ^T should be universal within a class.

The ϵ expansions for ξ_0^T/ξ_0 and for R_ξ^T are carried out in Appendix B, with the results

$$\left(\frac{\xi_0^T}{\xi_0}\right)^{d-2}=\frac{2^{1-2\beta}\epsilon}{K_d(n+8)}\left(1+\frac{17n+76}{2(n+8)^2}\epsilon\right)+O(\epsilon^3) \quad (43)$$

and

$$(R_\xi^T)^{(d-2)}=\frac{2^{\epsilon/4}\epsilon}{K_d^{2/d}(n+8)}\left(1+\frac{4n+2}{(n+8)^2}\epsilon\right)+O(\epsilon^3). \quad (44)$$

Extrapolated values of ξ_0/ξ_0^T and of R_ξ^T at $d=3$ are also exhibited in Table I. In these extrapolations we have used the explicit expressions (43) and (44), and substituted $\epsilon=1$, $d=3$, $K_d=K_3=1/2\pi^2$. These values may now be used, together with (42), to estimate ξ_T from measurements of C_-^s or of ξ_+.

As was the case for $T>T_c$, we can also calculate ξ_0^T/ξ_0 and R_ξ^T for general d in the limit $n\to\infty$. A direct extension of the calculation of Brézin and Wallace,[30] along lines similar to those presented in Appendix B, shows that in this limit, with the normalizations of Appendix B, $\rho_s=M^2$, and therefore

$$\left(\frac{\xi_0^T}{\xi_0}\right)^{d-2}=\frac{1}{K_dn}\,\frac{\sin\frac{1}{2}\pi\epsilon}{\frac{1}{2}\pi}+O\left(\frac{1}{n^2}\right), \quad (45)$$

in agreement with the large-n limit of (43). Thus, ξ_0^T/ξ_0 decreases monotonically to zero as n increases to infinity.

In order to obtain the large-n limit of R_ξ^T, it is now sufficient to know the large-n limit of A/A' and to use it together with (29) and (45). The ratio A/A' was recently calculated in this limit, by Abe and Hikami.[38] For $3<d<4$ they find

$A/A' = n[f(d) + O(1/n)]$, with $f(d)$ given by[38]

$$f(d) = \frac{2^{d(3-d)/(d-2)}\Gamma(d/2)\Gamma(2-d/2)}{\Gamma((4-d)/(d-2))\Gamma((2d-6)/(d-2))} \times \left(\frac{\Gamma(\frac{3}{2})\Gamma(d/2)}{\Gamma((d-1)/2)}\right)^{d/(d-2)} . \quad (46)$$

However, *at* $d=3$, A/A' is finite for $n\to\infty$, and becomes $A/A' = \pi^2/4 - 1 = 1.467\ldots$. Thus, the extrapolation of A/A' from these results to $n=2$, $d=3$ is quite ambiguous. We therefore prefer to use the ϵ expansions.

III. APPLICATION TO REAL SYSTEMS

In this section we wish to supplement the work of Stauffer, Ferer, and Wortis,[4] and of Ferer,[12] by a more complete discussion of two systems with continuous symmetry: superfluid helium ($n=2$) and the isotropic antiferromagnet $RbMnF_3$ ($n=3$).

A. Liquid helium

As mentioned in Eq. (40) and discussed more fully in Ref. 35, the stiffness constant ρ_s is proportional to the superfluid density $\tilde{\rho}_s$ (which has units of mass per unit volume). Thus the correlation length ξ_T may be obtained from the measured[39] superfluid density $\tilde{\rho}_s$ by the relation (for $d=3$)[36]

$$\xi_T = m_{He}^2 k_B T/\hbar^2 \tilde{\rho}_s . \quad (47)$$

Inserting the value[39] at SVP

$$\tilde{\rho}_s = 0.35(-t)^{0.67} \text{ g/cm}^3 , \quad (48)$$

we find, with $T=2.17$ K,

$$\xi_T = 3.57(-t)^{-0.67} \text{ Å} . \quad (49)$$

From $\tilde{\rho}_s$ measurements at higher pressure one also finds that Eq. (49) remains unchanged to within the accuracy of the absolute measurements (10%–20%).[39]

The correlation length ξ_+ above T_c cannot be measured, but we can use the series value of R_ξ^+ (Table I), combined with the experimental specific heat,[40] to arrive at a three-dimensional value of ξ_+. At SVP the specific heat has the asymptotic form (21) with[18]

$$A = 1.65\times 10^{22} \text{ cm}^{-3}, \quad A/A' = 1.065, \quad \alpha = -0.0154 . \quad (50)$$

Inserting the value $R_\xi^+ = 0.36$ from Table I into (17), and using (50), we thus find

$$\xi_+ = 1.41\, t^{-0.67} \text{ Å} \quad (51)$$

at SVP. Note that the ϵ-expansion value, $R_\xi^+ = 0.30$, will give a value of ξ_+ which will be smaller by ~20%. At higher pressures the absolute values of the specific heat are too uncertain to permit an independent calculation of ξ.[41] From the pressure independence of ξ_0^T, however, we are led to predict that *both* ξ_0 *and* A *will also be independent of pressure*.

The experimental value of A', Eq. (50), and the ϵ-expansion value of R_ξ^T, from Table I, may now be used to generate a theoretically predicted value of ξ_T, from Eq. (42). The result is

$$\xi_T = 3.8(-t)^{0.67} \text{ Å} , \quad (52a)$$

in surprisingly good agreement with the experimental result (49). An alternative way to predict ξ_T is to use ξ_+, Eq. (51), and the ϵ-expansion value of ξ_0/ξ_0^T from Table I. This yields

$$\xi_T = 5.2(-t)^{-0.67} \text{ Å}, \quad (52b)$$

which overestimates the experimental value by 50%. In view of the ambiguities in the extrapolation of the ϵ expansion, discussed in connection with the comparison to series results above T_c in Sec. II, we do not attach great significance to the discrepancy between (52b) and (52a).

It is interesting to note that if superfluidity were due to the condensation of *pairs* of helium atoms,[42] the appropriate mass to be inserted into (47) would be $2m_{He}$, leading to an experimental value of ξ_T larger by a factor of 4. Unfortunately, the uncertainties of the extrapolation of the ϵ expansion do not at present permit one to rule out this possibility definitively, although it seems quite unlikely. It is hoped that future, more accurate calculations might provide such a test.

The experimental results (49), (50), and (51) may also be used for a direct evaluation of the universal ratios R_ξ^+, R_ξ^T, and ξ_0/ξ_0^T. These are summarized in Table II, and are in reasonable agreement with the theoretical predictions of Table I. A universal ratio proportional to $(R_\xi^T)^d$ was first obtained from experimental data in the original work of Ferer.[12]

We should also mention that there exist other correlation lengths in superfluid helium, which arise in the analysis of the depression of T_λ in finite geometries, or the reduction of ρ_s in the neighborhood of a surface. These might more properly be called "healing lengths," since they measure the distance over which strong perturbations of the order parameter will decay. It is clear from scaling that the healing length ξ_H should have the usual exponent ν, and that its amplitude should be universally related to ξ_T, but the value of ξ_T/ξ_H is difficult to predict, since it depends on a detailed theory of surface effects near T_λ. Such effects are particularly difficult to describe for systems with continuous symmetry.[43] What is usually done in order to extract a value of ξ_H is to

TABLE II. Experimental values of ratios.

	Helium ($n=2$)	$RbMnF_3$ ($n=3$)
R_ξ^+	...	0.45[c]
R_ξ^T	0.90[a]	$1.8\sim 1.2$[d]
ξ_0/ξ_0^T	0.39[b]	$0.5\sim 0.7$[e]

[a] From Refs. 39 and 40.
[b] From Refs. 39, 40, and series value of R_ξ^+ in Table I.
[c] From Refs. 46 and 48.
[d] From Refs. 47 and 48.
[e] From Refs. 46 and 48.

apply mean-field theory,[44] with the simple boundary condition that the order parameter should vanish at the surface. Such an analysis was carried out by Ihas and Pobell,[45] who found the value $\xi_H = 1.2|t|^{-0.65}$ Å. More importantly, however, these authors were able to check directly the universality of ξ_H/ξ_T as a function of pressure, by combining their measurements of the depression of T_λ in finite geometries, with bulk measurements of ρ_s.

Finally, we remark that although two nonuniversal scale factors are needed to specify the critical behavior completely (in addition to the universal exponents and amplitude ratios), a certain amount of information is already available by determining a single amplitude, say A. This is because there is a subset of "hydrodynamic" exponents and amplitudes which are related among themselves, independently of the others. These are the exponents α and ν, and the amplitudes A, A', and ξ_0^T. The remaining "microscopic" exponents and amplitudes are as yet inaccessible to experiment in liquid helium. Thus there is no quantitative information available on the strength of the Bose condensate,[29] which is given by the nonuniversal amplitude B. Another "hydrodynamic" exponent in liquid helium is the dynamic exponent[32,33] z, which is related by a scaling law to α and ν, with an associated universal amplitude ratio R_λ, discussed in Ref. 35.

B. $RbMnF_3$

In the isotropic antiferromagnet $RbMnF_3$ there are neutron scattering measurements of ξ_+,[46] and of the spin-wave velocity,[47] and thermodynamic measurements of the specific-heat amplitudes A and A'.[48] The measured value[46] of $\xi_+ = \kappa_+^{-1}$ is

$$\xi_+ = 2.1\, t^{-0.71} \text{ Å}, \tag{53}$$

whereas the spin-wave velocity c is given by[47]

$$\hbar c = 25.3(-t)^{0.27} \text{ meV Å}. \tag{54}$$

As explained in Sec. V C of Ref. 35, the correlation length $\xi_T = (\kappa_-)^{-1}$ is related to c by

$$\xi_T = k_B T_N (g\mu_B)^2/\hbar^2 c^2 \tilde{\chi}, \tag{55}$$

where[49] $\tilde{\chi} = 3.9\times 10^{-4}$ is the macroscopic susceptibility per cm^3, in electromagnetic units, $g\mu_B = 2\times 0.927\times 10^{-20}$ ergs/g, and $T_N = 83.02$ K. Combining Eqs. (54) and (55) we find

$$\xi_T = 6.2(-t)^{-0.54} \text{ Å}. \tag{56}$$

Clearly the relation $\nu = \nu'$ is only obeyed very approximately by these fits.

The specific heat was found to be of the form (10) and (21), with[18,48]

$$A = 9.87\times 10^{21} \text{ cm}^{-3}, \quad A/A' = 1.46, \quad \alpha = -0.135. \tag{57}$$

We may use the experimental values (57) for A and A' and the ϵ-expansion (or series) values for R_ξ^+ and R_ξ^T to generate theoretically predicted values of ξ_+ and of ξ_T:

$$\begin{aligned} \xi_+ &= 1.68\, t^{-0.71} \text{ Å} \quad (\epsilon \text{ expansion}), \\ \xi_+ &= 1.96\, t^{-0.71} \text{ Å} \quad (\text{series}), \end{aligned} \tag{58}$$

and

$$\xi_T = 4.65(-t)^{-0.71} \text{ Å}. \tag{59a}$$

The result (58) again shows that the extrapolated ϵ expansion underestimates ξ_+ by ~15%. Considering the error bars, the series prediction is in good agreement with the experimental result (53). Since the experimental fit (56) does not agree with the scaling relation $\nu = \nu'$, it is difficult to compare (59) with (56). In the typical experimental range probed in Ref. 46, $|t| \sim 10^{-1} \sim 10^{-2}$, the theoretical value (59a) overestimates the experimental fit (56) by 10%–60%.

An alternative predicted estimate of ξ_T follows from the experimental value of ξ_+, (53), and the ϵ-expansion extrapolation for ξ_0/ξ_0^T (Table I). This yields

$$\xi_T = 7.0\, t^{-0.71} \text{ Å}, \tag{59b}$$

which again overestimates (56) (for $|t| \sim 10^{-1}$–10^{-2}) by roughly 60%. We thus see that the ϵ expansion seems to overestimate ξ_0^T for $n = 3$, as was also found in the case of helium, $n = 2$.

We could also evaluate the experimental ratios R_ξ^+, R_ξ^T, and ξ_0/ξ_0^T directly from Eqs. (53), (56), and (57). The violation of the relation $\nu = \nu'$ in (53) and (56) then leads to a temperature dependence of R_ξ^T and ξ_0/ξ_0^T,

$$R_\xi^T = 2.6|t|^{0.17}, \tag{60}$$

$$\xi_0/\xi_0^T = 0.34|t|^{-0.17}. \tag{61}$$

Substituting $|t| \sim 10^{-1}$–10^{-2} gives the numbers presented in Table II. In view of the strong temperature dependences which occur in (60) and (61) it is difficult to draw conclusions about the accuracy of the ϵ expansion in this case. It is hoped that more definitive experimental investigations will be made on a magnetic system with continuous symmetry.

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APPENDIX A: RENORMALIZATION-GROUP PROOF OF TWO-SCALE-FACTOR UNIVERSALITY

We start with a reduced Ginzburg-Landau-Wilson Hamiltonian,[5]

$$\mathcal{H}_0(\vec{\psi}) = -\frac{1}{2}\int (r_0|\vec{\psi}|^2 + |\nabla\vec{\psi}|^2 + 2u_0|\vec{\psi}|^4 + \cdots)\, d^dx$$
$$= -\frac{1}{2}\int_{\vec{q}} (r_0+q^2)\vec{\psi}_{\vec{q}}\cdot\vec{\psi}_{-\vec{q}}$$
$$+u_0\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\vec{\psi}_{\vec{q}}\cdot\vec{\psi}_{\vec{q}'}\vec{\psi}_{\vec{q}''}\cdot\vec{\psi}_{-\vec{q}-\vec{q}'-\vec{q}''}+\cdots, \tag{A1}$$

where $\int_{\vec{q}}$ means $(2\pi)^{-d}\int d^dq$, with $|\vec{q}|<\Lambda=O(1)$. For convenience, we set $\Lambda=1$ in this appendix. We then construct recursion relations for r_l, u_l, etc., and rescale the order parameter after each iteration by a factor ζ_l so that the coefficient of $|\nabla\psi|^2$ remains equal to 1.[5]

The free energy may be written as[25]

$$F(r_0, u_0, \ldots) = nK_d\int_0^\infty \mathcal{G}(l)e^{-dl}\,dl\,, \tag{A2}$$

where

$$\mathcal{G}(l) = G(r_l, u_l, x_l) + (\tfrac{1}{2}\eta_l - 1)/d\,. \tag{A3}$$

Here, x_l represents the fast transients (e.g., the coefficients of $|\vec{\psi}|^6$ or of $|\nabla^2\vec{\psi}|^2$), and may be set equal to zero for large l. The quantity η_l is related to the spin rescaling factor at the lth iteration,

$$\zeta_l = e^{(1-\eta_l/2)\delta l}, \tag{A4}$$

where $e^{\delta l}$ is the change in the length scale, and

$$G(r_l, u_l, 0) = \tfrac{1}{2}\ln(1+r_l) + O(u_l)\,. \tag{A5}$$

At T_c (or $r_0 = r_{0c}$), for large l,

$$r_l \to r^*,\quad u_l\to u^*,\quad x_l\to 0,$$
$$\eta_l\to\eta,\quad G\to G^*, \tag{A6}$$

etc., where η is the usual critical-correlation exponent. For r_0 slightly above r_{0c}, we now define L such that $r_L=1$. The correlation length is then given by $\xi = e^L[1+O(u_L^2)]$, since there are no q-dependent contributions to the correlation function $\hat{\chi}(q, t, 0)$ at order u_L.[8] Let N now be a value of l sufficiently large so that for $l\geq N$ all "transients" x_l have died away and are negligible. For finite $\epsilon>0$, (u_l-u^*) can also be considered a transient. We assume that $r_0 - r_{0c}$ is sufficiently small, so that $L\gg N$. Then, for $l>N$, all the variables (r_l, η_l, etc.) are functions only of $l-L$. In particular, apart from an additive nonuniversal constant, $\mathcal{G}(l)$ is also a function of $l-L$. Furthermore, we may then expand quantities in powers of $e^{-(L-l)/\nu}$:

$$r_l \simeq r^* + r_1e^{-(L-l)/\nu} + r_2e^{-2(L-l)/\nu} + \bar{r}(l-L)\,,$$
$$\eta_l = \eta + \eta_1e^{-(L-l)/\nu} + \eta_2e^{-2(L-l)/\nu} + \bar{\eta}(l-L)\,,$$
$$G(r_l, u_l, x_l) + \tfrac{1}{2}\eta_l/d \equiv \varphi(l-L) \tag{A7}$$
$$= G^* + \tfrac{1}{2}\eta/d + \varphi_1e^{-(L-l)/\nu} + \varphi_2e^{-2(L-l)/\nu} + \bar{\varphi}(l-L)\,,$$

etc. We can now divide the integral in (A2) into three parts,

$$\int_0^\infty = \int_0^N + \int_N^L + \int_L^\infty\,. \tag{A8}$$

The contribution from the last term, $-1/d$, in (A3) is a constant, independent of r_0, and may be dropped. The contribution from the first integral in (A8) is nonuniversal, but analytic in r_0, and thus does not affect the singular term in the free energy, F_s. The last term in (A8) is of the form

$$\int_L^\infty\left(G(r_l, u_l, 0) + \frac{1}{2}\frac{\eta_l}{d}\right)e^{-dl}\,dl = a_1e^{-dL}, \tag{A9}$$

where

$$a_1 = \int_0^\infty \varphi(\tau)e^{-d\tau}\,d\tau \tag{A10}$$

is universal. Similarly,

$$\int_N^L [G(r_l, u_l, 0) + \tfrac{1}{2}\eta_l]$$
$$= \left(G^* + \frac{1}{2}\frac{\eta}{d}\right)\frac{e^{-dN}-e^{-dL}}{d} + \varphi_1\frac{e^{-(d-1/\nu)N}e^{-L/\nu} - e^{-dL}}{d-1/\nu}$$
$$+\varphi_2\frac{e^{-(d-2/\nu)N}e^{-2L/\nu}-e^{-dL}}{d-2/\nu} + a_2e^{-dL}\,, \tag{A11}$$

where

$$a_2 = \int_{-\infty}^0 \bar{\varphi}(\tau)e^{-d\tau}\,d\tau < \infty \tag{A12}$$

is again universal. The first term in (A11) combines with the corresponding one in (A9) to give a contribution independent of L. The terms proportional to $e^{-L/\nu}$ and $e^{-2L/\nu}$ are regular, since $e^{-L/\nu}\propto t$ ($\xi\simeq e^L$). Hence, the only singular terms are proportional to e^{-dL}, and the coefficient is a universal constant,

$$F_s = nK_d a e^{-dL} = nK_d a \xi^{-d} , \qquad \text{(A13)}$$

where

$$a = a_1 + a_2 - \frac{G^* + \frac{1}{2}\eta}{d} - \frac{\varphi_1}{d - 1/\nu} - \frac{\varphi_2}{d - 2/\nu} \qquad \text{(A14)}$$

is universal. This completes our proof. Note that if $d\nu = 2$ then the last term in (A14) corresponds to logarithms.[24] In the limit $\epsilon \to 0$, only this last term remains, and we have

$$a = \varphi_2 \nu / \alpha , \qquad \text{(A15)}$$

with, to lowest order,

$$\varphi_2 = \frac{1}{2}\frac{d^2}{dr^2}\left[\tfrac{1}{2}\ln(1+r)\right]_{r=0} = -\tfrac{1}{4}. \qquad \text{(A16)}$$

Using $\nu = \frac{1}{2} + O(\epsilon)$, we thus identify

$$\alpha F_s \xi^d = -\tfrac{1}{8} n K_d + O(\epsilon), \qquad \text{(A17)}$$

in agreement with the lowest-order result (24) and with Ref. 24.

Note added in proof. We have been informed by Professor F. Wegner that he has independently proved two-scale-factor universality using the renormalization group.[51]

APPENDIX B: ϵ EXPANSION OF $R_\xi^\pm$ AND R_ξ^T

The basic Hamiltonian we use is again that of Eq. (A1), with the addition of a magnetic field term. For this Hamiltonian, Brézin, Wallace, and Wilson[6] derived the equation of state (4). For general cutoff Λ, their result is[50]

$$\begin{aligned} h(x) = {} & \Lambda^{-[(n+2)\epsilon/(n+8)]} x + 4u_0 \Lambda^{-\epsilon} \\ & + \frac{\epsilon}{2(n+8)}\left[3(x+12u_0)\ln(x+12u_0)\right. \\ & \left. + (n-1)(x+4u_0)\ln(x+4u_0)\right] + O(\epsilon^2), \end{aligned} \qquad \text{(B1)}$$

where one must substitute

$$4K_d u_0 = \frac{\Lambda^\epsilon \epsilon}{n+8}\left(1 + \frac{9n+42}{(n+8)^2}\epsilon\right) + O(\epsilon^3) \qquad \text{(B2)}$$

in order to obtain the correct exponentiations of M in (4).

From (B1) we immediately identify

$$x_0 = (2/\Lambda^2)^{3\epsilon/(n+8)}(4u_0)^{1+3\epsilon/(n+8)} + O(\epsilon^3), \qquad \text{(B3)}$$

and

$$h_0 = \Lambda^{-\epsilon} 3^{9\epsilon/2(n+8)}(4u_0)^{1+\epsilon/2} + O(\epsilon^3). \qquad \text{(B4)}$$

As noted in Ref. 8, the ϵ expansion of the two-spin correlation function for the Hamiltonian (A1) is of the form

$$\hat{\chi}(q, t, 0) = \frac{1}{r + q^2} + O(\epsilon^2), \qquad \text{(B5)}$$

where r is the exact inverse susceptibility,

$$\hat{\chi}(0, t, 0) \equiv 1/r. \qquad \text{(B6)}$$

Thus we identify the correlation length ξ [see Eq. (11)] as

$$\begin{aligned} \xi^2 &= 1/r + O(\epsilon^2) \\ &= \hat{\chi}(0, t, 0) + O(\epsilon^2) = \Gamma t^{-\gamma} + O(\epsilon^2). \end{aligned} \qquad \text{(B7)}$$

Since $\gamma = (2-\eta)\nu = 2\nu + O(\epsilon^2)$, we immediately identify

$$\xi_0^2 = \Gamma + O(\epsilon^2). \qquad \text{(B8)}$$

The amplitude Γ is readily calculable from the equation of state (B1), using Eq. (9). The result is

$$\Gamma = \Lambda^{(n+2)\epsilon/(n+8)} + O(\epsilon^2). \qquad \text{(B9)}$$

The specific-heat amplitude A may be obtained by integrating the equation of state,[15,16]

$$\begin{aligned} A &= h_0 x_0^{\alpha-2}\tilde{A} \\ &= h_0 x_0^{\alpha-2}\beta\alpha(1-\alpha)\int_0^\infty dy\, y^{\alpha-3} \\ &\quad \times[\tilde{h}(y) - \tilde{h}(0) - \tilde{h}'(0)y - \tfrac{1}{2}\tilde{h}''(0)y^2]. \end{aligned} \qquad \text{(B10)}$$

The universal amplitude $\tilde{A}$ can be found from the explicit ϵ expansion[6] of $\tilde{h}(\tilde{x})$ as[22]

$$\tilde{A} = \frac{n\epsilon}{4(n+8)}\left[1 + \epsilon\left(1 + \frac{3\ln(16/27)}{2(n+8)} - \frac{30}{(n+8)^2}\right)\right] + O(\epsilon^3). \qquad \text{(B11)}$$

Using the scaling relation (15) we may now express the ratio R_ξ^+, Eq. (16), as

$$(R_\xi^+)^d = \xi_0^d h_0 x_0^{\alpha-2}\tilde{A}. \qquad \text{(B12)}$$

Combining Eqs. (B3), (B4), (B8), and (B9), and using the ϵ expansion of the exponents,[5] we find the universal expression

$$h_0 x_0^{\alpha-2}\xi_0^d = K_d(n+8)\epsilon^{-1}\left[1 - \epsilon\left(\frac{9n+42}{(n+8)^2} + \frac{3\ln(16/27)}{2(n+8)}\right) + O(\epsilon^2)\right]. \qquad \text{(B13)}$$

This result coincides, up to a factor of d,[23] with the quantity calculated in Ref. 10.

We may now combine Eqs. (B11)–(B13) to find the result for R_ξ^+ quoted in Eq. (24).

The expansion of the transverse correlation function at finite q proceeds in a manner analogous to the calculation of Brézin, Wallace, and Wilson,[6] who considered the case $q = 0$, $H \neq 0$. The first-order result at $H = 0$, $t < 0$, $q \to 0$ is

$$\hat{\chi}_T^{-1}(\vec{q}, t, M) = q^2 - 64u_0^2 M^2 \int \frac{d^4p}{(2\pi)^4} \times\left(\frac{1}{r_L + (\vec{p}+\vec{q})^2} - \frac{1}{r_L + p^2}\right)p^{-2}, \qquad \text{(B14)}$$

where r_L is the inverse zero-momentum longitudinal susceptibility, to lowest order in ϵ, and is given by

$$r_L = r_0 + 12u_0 M^2 + O(\epsilon) = 8u_0 M^2 + O(\epsilon). \qquad \text{(B15)}$$

The second step in (B15) follows from Eq. (B1), which at zero field yields $r_0 + 4u_0 M^2 = O(\epsilon)$. Expanding the integral in (B14) for small q^2, and using (B2) and (B15), we immediately find (at $H=0$)

$$\hat{\chi}_T^{-1}(\vec{q}, t, M) = q^2\left(1 + \frac{\epsilon}{2(n+8)} + O(\epsilon^2)\right) + O(q^4). \qquad \text{(B16)}$$

The correlation length ξ_T, Eq. (38), is thus[50]

$$\xi_T^{d-2} = M^{-2}\left(1 - \frac{\epsilon}{2(n+8)} + O(\epsilon^2)\right) = x_0^{-2\beta}\left(1 - \frac{\epsilon}{2(n+8)} + O(\epsilon^2)\right)(-t)^{-2\beta}. \qquad \text{(B17)}$$

Inserting the expressions (B3), (B8), and (B9) we thus find

$$\left(\frac{\xi_0^T}{\xi_0^+}\right)^{d-2} = (8u_0)\left(1 - \frac{\epsilon}{2(n+8)}\right)2^{-2\beta}\Lambda^{-\epsilon}. \qquad \text{(B18)}$$

Using (B2) we arrive at (43). Finally, the value of R_ξ^T quoted in (44) is obtained by inserting (B3), (B11), (B17), and (26) into (42).

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[19]The parameter $(R_\xi^+)^d$ was denoted X in Ref. 4; it is related to X of Ref. 10 by $(R_\xi^+)^d = X\tilde{A}$, where $\tilde{A}$ is a universal quantity defined in Eq. (B10).

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