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# Renormalization-Group Calculations of Divergent Transport Coefficients at Critical Points

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Recursion relations correct to lowest order in $\epsilon = 4 - d$ are obtained for simple models representing the critical dynamics of a binary liquid, a Heisenberg antiferromagnet, superfluid helium, and related systems. The exponents for the divergences of the transport coefficients satisfy the "scaling" relations predicted by mode-mode coupling theory. The viscosity of the binary liquid is predicted to diverge as $\kappa^{-\epsilon/19}$ near $d = 4$, which is a new result.

Recently the renormalization-group method[1] has been employed to calculate dynamic critical exponents for time-dependent Ginzburg-Landau models with Hermitian Liouville operators.[2] As pointed out by Kawasaki and by Kadanoff and Swift,[3] the transport coefficients in such models cannot diverge; on the other hand, more general models, with non-Hermitian Liouville operators, may possess divergences in their transport coefficients, brought about by the nonlinear couplings of their hydrodynamic modes near the critical point.[4,5] We have constructed two simple models of the latter type—the first representing the phase-separation point of a binary fluid or the critical point of a pure fluid, and the second a "planar ferromagnet," a Heisenberg antiferromagnet, or superfluid helium. Analysis of these models using the renormalization-group method and the $\epsilon$ expansion[1,2] yields divergent transport coefficients and scaling relations, which agree in most respects with the predictions of the mode-mode coupling approach of Kawasaki,[4] Kadanoff and Swift,[5] and others. In addition we have obtained values for the exponents of the transport coefficients in the binary liquid near four dimensions, which are new results.

Model I is defined by the equations

$$\frac{\partial \psi}{\partial t} = \lambda_0 \nabla^2 \frac{\delta F}{\delta \psi} - g_0 \nabla \psi \cdot \frac{\delta F}{\delta \vec{j}} + \theta, \tag{1a}$$

$$\frac{\partial \vec{j}}{\partial t} = \left( \eta_0 \nabla^2 \frac{\delta F}{\delta \vec{j}} + g_0 \nabla \psi \frac{\delta F}{\delta \psi} + \vec{\zeta} \right)_{TP}, \tag{1b}$$

$$F = \int d^d x \left( \tfrac{1}{2} r_0 \psi^2 + \tfrac{1}{2} |\nabla \psi|^2 + u_0 \psi^4 + \tfrac{1}{2} j^2 \right), \tag{1c}$$

where $\psi(\vec{x}, t)$ is a scalar order parameter, $\vec{j}(\vec{x}, t)$ is the transverse part (TP) of a $d$-component vec-

tor field, and $\lambda_0$, $g_0$, $\bar{\eta}_0$, $r_0$, and $u_0$ are constants. The functions $\theta$ and $\vec{\zeta}$ are Langevin noise sources satisfying the relations

$$\langle \theta(\vec{x}, t)\theta(\vec{x}', t')\rangle = -2\lambda_0\nabla^2\delta(\vec{x}-\vec{x}')\delta(t-t'), \quad (2a)$$

$$\langle \zeta_\alpha(\vec{x}, t)\zeta_{\alpha'}(\vec{x}', t')\rangle = -2\bar{\eta}_0\nabla^2\delta(\vec{x}-\vec{x}')\delta(t-t')\delta_{\alpha\alpha'}. \quad (2b)$$

Fluctuations in $\psi$ and $\vec{j}$ are confined to wave vectors $k$ less than a cutoff $\Lambda$.

The above model, with $\psi$ denoting concentration fluctuations and $\vec{j}$ the momentum density, is a simple representation of a binary liquid, which is incompressible, symmetric about the critical concentration, and neglects heat flow. It is equivalent near $T_c$ to the model in Eqs. (3.25a) and (3.25b) of Ref. (4). The parameters $\bar{\eta}_0$ and $\lambda_0$ represent "bare" values of the viscosity and of the transport coefficient for concentration fluctuations. The mass density and the temperature have been set equal to unity, and $g_0 = 1$ in these units. The aim of the present investigation is to calculate the exponents for the $k$-dependent "physical" transport coefficients $\lambda \sim k^{-x_\lambda}$, $\bar{\eta} \sim k^{-x_{\bar{\eta}}}$ at $T_c$, for $k \ll \Lambda$. For $T > T_c$, $k$ is replaced by $\kappa$, the inverse correlation length. The equilibrium distribution of $\psi$ and $\vec{j}$ is proportional to $e^{-F}$. Since $\psi$ and $\vec{j}$ are uncoupled in $F$, the equilibrium properties are clearly the same as for the Ginzburg-Landau-Wilson model.[1]

We have constructed recursion relations[1] for (1) under the action of the dynamic renormalization group.[2,6] These relations, which we believe to be correct to first order in $\epsilon \equiv 4-d$, are given, at $T_c$, by

$$\lambda_{l+1} = b^{z-4}(\lambda_l + \bar{\eta}_l^{-1} g_l^2 \tfrac{3}{4} K \ln b), \quad (3a)$$

$$\bar{\eta}_{l+1} = b^{z-2}(\bar{\eta}_l + \lambda_l^{-1} g_l^2 \tfrac{1}{24} K \ln b), \quad (3b)$$

$$g_{l+1} = b^{z-3+\epsilon/2} g_l, \quad (3c)$$

where $b$ is the dilation factor for the length scale, $b^z$ is the scaling factor for frequencies, the scaling factors for $\vec{j}$ and $\psi$ are chosen to preserve the second and fourth terms in (1c), and $K = (8\pi^2)^{-1}$. The recursion relations (3) are derived[6] from a diagrammatic expansion in the "vertices" $g_0$ and $u_0$, such as the one described by Martin, Siggia, and Rose.[7] The relevant diagrams to first order in $\epsilon$ are similar to those leading to Eqs. (4.56a) and (4.56b) of Kawasaki,[4] and do not involve $u_0$.

The three recursion relations may be reduced to one equation for the ratio $f_l \equiv g_l^2/\bar{\eta}_l\lambda_l$, whose fixed point is $f^* = \frac{24}{19}K^{-1}\epsilon$. The condition for Eq. (3a) to have a fixed point $\lambda^*$ is then

$$z - 4 = -\tfrac{3}{4}Kf^* = -\tfrac{18}{19}\epsilon. \quad (4)$$

But $z$ is the characteristic exponent for the order-parameter relaxation rate[2]

$$\omega_\psi = (\lambda/\chi_\psi)k^2 \sim \lambda k^{4-\eta} \sim k^z,$$

where $\eta$, the usual exponent for the order-parameter static susceptibility $\chi_\psi(k)$, is of order $\epsilon^2$. It follows that

$$x_\lambda = \tfrac{18}{19}\epsilon + O(\epsilon^2). \quad (5)$$

By transforming the scaled recursion relations (3b) and (3c) back to the original physical scale, one also finds the exponent for the viscosity,

$$x_{\bar{\eta}} = \tfrac{1}{19}\epsilon + O(\epsilon^2). \quad (6)$$

The exponent values in (5) and (6) represent new results, which disagree with the statement in Ref. 4 (p. 41) that $\bar{\eta}$ remains finite at $T_c$. The practical significance of our result for $x_{\bar{\eta}}$ is difficult to assess, however, since terms of higher order in $\epsilon$ may change the value significantly for $d = 3$, and present-day experiments do not distinguish between a slightly positive or slightly negative exponent.[8]

The mode-mode theory[4,5] leads to a general relation for the transport coefficients,[9] which may be written, for arbitrary $d$, as a "scaling relation" between exponents, namely

$$x_\lambda + x_{\bar{\eta}} = \epsilon - \eta. \quad (7)$$

[Note that $x_{\bar{\eta}}$ is the exponent for the *ordinary* (low-frequency) viscosity.] Equation (7) is obeyed by the values obtained in (5) and (6). Moreover, a preliminary analysis of corrections to the recursion relations (3) of higher order in $\epsilon$, similar to that given by Wilson and Kogut for the static properties,[1] indicates that (7) will hold *to all orders in* $\epsilon$, provided that no unforeseen divergences arise in the theory.[6]

Our second model (planar ferromagnet[10]) may be written as

$$\frac{\partial\psi}{\partial t} = -2\Gamma_0\frac{\delta F}{\delta\psi^*} - ig_0\psi\frac{\delta F}{\delta m} + \theta, \quad (8a)$$

$$\frac{\partial m}{\partial t} = \lambda_0\nabla^2\frac{\delta F}{\delta m} + 2g_0\,\mathrm{Im}\,\psi^*\frac{\delta F}{\delta\psi^*} + \zeta, \quad (8b)$$

$$F = \int d^dx\,(\tfrac{1}{2}r_0|\psi|^2 + \tfrac{1}{2}|\nabla\psi|^2 + u_0|\psi|^4 + \tfrac{1}{2}m^2), \quad (8c)$$

$$\langle\theta(\vec{x}, t)\theta^*(\vec{x}', t')\rangle = 4\Gamma_0\delta(\vec{x}-\vec{x}')\delta(t-t'), \quad (8d)$$

$$\langle\zeta(\vec{x}, t)\zeta(\vec{x}', t')\rangle = -2\lambda_0\nabla^2\delta(\vec{x}-\vec{x}')\delta(t-t'). \quad (8e)$$

Here the order parameter $\psi(\vec{x}, t)$ is a complex field, representing the (nonconserved) $x$ and $y$ components of the magnetization, while $m(\vec{x}, t)$ is the $z$ component of magnetization, which is conserved. [As in (1), the heat-flow mode has been neglected.] Recursion relations may be derived for this model, as before. The dynamic critical exponent $z$ for the order parameter, and the exponent for the divergence of the transport coefficient $\lambda$, are found to be

$$z = 2 - x_\lambda = 2 - \tfrac{1}{2}\epsilon = d/2. \quad (9)$$

Model II may also be extended to the case of a Heisenberg antiferromagnet, where $\psi$ and $m$ each have three real components, and the same results (9) are obtained. Equation (9), which coincides with the result of mode-mode coupling [Eq. (4.47) of Ref. 4] and of dynamic scaling[11] for these models for arbitrary $d$, appears to be correct to all orders in $\epsilon$, on the basis of renormalization-group considerations.[6] The result (9) also agrees with a microscopic analysis of the quantum-mechanical Heisenberg antiferromagnet, carried out in the limit of long-range forces, where the static properties are those of the Gaussian model.[12]

Equations (8) possess a symmetry between $m$ and $-m$ appropriate to a planar ferromagnet in zero external field. A more general model, appropriate to the case of finite field in the $z$ direction, has a term in $F$ proportional to $m|\psi|^2$, and also includes the possibility of an imaginary part to the coefficient $\Gamma_0$ in (8a). We have developed recursion relations for such a model, and we find for the dynamic exponents

$$z = 2 - x_\lambda + \alpha/\nu = \tfrac{1}{2}(d + \alpha/\nu), \quad (10)$$

where $\alpha$ and $\nu$ are the usual exponents for the specific heat and correlation length, and $\alpha/\nu = \frac{1}{5}\epsilon + O(\epsilon^2)$. These results again coincide with the predictions of dynamic scaling[11] and of mode-mode coupling[4,5] [e.g., Eq. (4.45) of Ref. 4] for this system, which state that (10) should hold if $\alpha > 0$, while (9) is valid for $\alpha < 0$.

The asymmetric planar magnet is a simplified model for the order-parameter and heat-flow modes of $He^4$ near its $\lambda$ transition.[10,11,13] The variable $m$ is then proportional to fluctuations in the *energy* of the liquid and $\lambda$ is the thermal conductivity. A more complete model for helium should also include the separately conserved density and momentum fields (cf. model I) which lead to first-sound and viscous-relaxation modes in the hydrodynamics. On the basis of dynamic scaling[13,11] and mode-mode analyses,[4,5] however, it is believed that the simplified model already gives the correct exponents $z$ and $x_\lambda$.

Attempts by a number of authors[14,15] to derive the critical dynamics of superfluid helium starting from the microscopic Hamiltonian for a Bose liquid have so far not led to a derivation of the scaling result (10) or (9). Most recently,[15] an attempt to extract the dynamic critical exponent $z$ for the Bose fluid directly from the microscopic perturbation expansion for the self-energy, using the same diagrams as are necessary to calculate the static exponent $\eta$ to order $\epsilon^2$, has yielded results in disagreement with (10). We believe this microscopic calculation to be incorrect because it fails to treat the conservation laws and hydrodynamic modes in a proper manner. Specifically, since the interaction is taken to zero as $d \to 4$, the mean free path of the bosons becomes infinite, and the range of validity of hydrodynamics is confined to smaller and smaller $k$. Presumably, the true dynamic critical behavior is also confined to very small $k$, and it may not be possible to extract critical exponents from a matching condition on the $\ln k$ terms in a perturbation series which holds for $\epsilon \to 0$, at fixed $k$. The various parameters entering the recursion relations for the asymmetric planar ferromagnet in fact correspond to five "slow transients," whose coefficients must be properly adjusted in order to obtain the correct answer from a matching condition.

From another point of view, if one attempts to construct recursion relations directly for the dynamics of the Bose Hamiltonian, similar to those discussed here or in Ref. 2, one immediately encounters a difficulty: After the first iteration, the interaction $u_l$ acquires a singular dependence on frequency and wave vector as $k \to 0$. This problem was also discussed in Ref. 2, in connection with the classical dynamics of a system of coupled anharmonic oscillators, a model for a displacive transition in a solid. The philosophy of the mode-mode approach suggests that the appropriate way to eliminate these difficulties is to introduce explicitly an auxiliary dynamic field, such as $\vec{j}$ or $m$ in the present paper or the energy density in Ref. 2, and to develop the renormalization group for the order parameter and the auxiliary field simultaneously. The fact that the calculations of the present paper confirm relations (7), (9), and (10) may be taken as a consistency check on the mode-mode philosophy.[4,5]

The multicomponent Bose gas, which has been discussed recently by a number of authors,[16] differs in certain respects from the helium case, and will be considered in a separate paper.

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# Observation of the Critical Velocity Peak in Superfluid Films*

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Critical flow velocities measured by Doppler-shifted third sound in unsaturated helium II films are shown to exhibit a well-defined maximum as a function of film thickness and at this maximum to be considerably higher than those usually found in film flow. In addition the critical velocity is strongly temperature dependent below the maximum and relatively temperature independent above.

It is a well-known observation that the critical flow velocity in superfluid helium increases as the channel width or film thickness, $d$, decreases.[1,2] Many relations such as $V_{sc}=d^{-1/4}$ cgs (the Leiden critical velocity)[2] and $V_{sc}=\hbar/md$ (the Feynman critical velocity)[3] have been suggested in the literature. On the other hand, when $d$ becomes so small that it approaches the onset thickness for superflow, $V_{sc}$ must vanish.[4] These observations imply that at some intermediate thickness $V_{sc}$ has a maximum value, a result which, while recognized, seems never to have been explicitly stated in the literature and certainly never directly observed. Using Doppler-shifted third sound in unsaturated films we present here the first direct observations of this maximum and find it can be considerably higher than values usually associated with film flow.

The experimental apparatus is shown schematically in Fig. 1 and described briefly below.[5] An unsaturated film is condensed onto the tubular glass substrate which has outside diameters of 0.395 and 1.625 in. at the top and bottom sections, respectively. The surface of the entire glass substrate was flamed smooth and a nonsealing soap fillet provides a smooth transition from the glass surface to the copper housing. A heater at the top drives the film up from the larger bottom section. Evaporation from the heater, recondensation on the copper walls, and flow back to the bottom section complete the cycle. A supply of $Al_2O_3$ powder (500-Å grain size) acts as a film